GX 6x2154

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

Semester One Examinations, 2003/2004

Exam Code(s)	4CS2, 4FM2
Exam(s)	Fourth Science
Module Code(s)	MP491
Module(s)	Non-Linear Systems
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan
Internal Examiner(s)	Dr. M. Ó Confhaola
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Instructions: Answer: THREE questions.
All questions carry the same marks.

Duration	2 hours
No. of Answer books	1

Requirements

Handout	
MCQ	
Statistical Tables	
Graph paper	Yes
Log Graph Paper	
Other Material	

No. of Pages	3
Department(s)	Mathematical Physics

1. a. The one-parameter family of non-linear ordinary differential equations

$$\frac{dx}{dt} = 2x^3 - ax$$

where a is a real paranmeter, is characterised by a bifurcation. Investigate the nature of the bifurcation, identify its type and sketch the corresponding bifurcation diagram.

b. Classify the equilibrium point, at $(0,0)$, of the following two-dimensional system of linear ordinary differential equations

$$\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = 3x - 2y \ .$$

Using *either* the method of isoclines *or* the Jordan normal form approach, sketch the xy-phase portrait for this problem.

c. The two-dimensional system of non-linear ordinary differential equations

$$\frac{dx}{dt} = a\,x + y + x\,(3x^2 + 2y^2)\,, \quad \frac{dy}{dt} = -x + a\,y + y\,(3x^2 + 2y^2)$$

possesses a limit cycle solution for certain values of the parameter a. Investigate the nature of the Hopf bifurcation that occurs at the critical value of a and identify what that critical value is.

2. Consider the two-dimensional system of non-linear ordinary differential equations

$$\frac{dx}{dt} = x - y + xy\,, \quad \frac{dy}{dt} = -x - y\ .$$

a. Locate and classify all the equilibrium points in the associated phase plane.

b. Explain *briefly* the motivation for the method of isoclines.

c. Identify, and sketch separately, the isoclines corresponding to $k = 0, 1, -1$ and ∞.

d. By using all four isoclines together, sketch a sufficient number of phase plane curves to illustrate the phase plane portrait for this problem.

3. a. Explain what is meant by (*i*) the index of an equilibrium point and (*ii*) a limit cycle for a two-dimensional system of non-linear ordinary differential equations.

b. State both the Index theorem and the Poincaré-Bendixson theorem.

c. By considering the solution path directions across the topographic system of concentric circles

$$x^2 + y^2 = \text{constant}$$

show, using the Poincaré-Bendixson theorem, that there is a limit cycle for the system

$$\frac{dx}{dt} = 2x + 2y - x\left(\frac{1}{2}x^2 + y^2\right), \quad \frac{dy}{dt} = -2x + y - y\left(x^2 + \frac{3}{2}y^2\right)$$

and that this limit cycle is constrained within the annulus

$$\sqrt{\frac{2}{3}} \le \sqrt{x^2 + y^2} \le 2\ .$$

You may assume that $(0,0)$ is the only equilibrium point of the system of equations.

4. The one-parameter family of one-dimensional difference equations

$$x_{n+1} = ax_n - (x_n)^3$$

is obtained from the map

$$f(x) = ax - x^3$$

by iteration.

a. Investigate for what values of the parameter a the equation has fixed points, what those fixed points are and their stability properties.

b. Investigate for what values of a the above family of equations has period-2 points, and show that if p is a period-2 point then it occurs as a root of the equation

$$x(x^2 - a + 1)(x^2 - a - 1)(x^4 - ax^2 + 1) = 0 \ .$$

Hence, or otherwise, identify all the period-2 orbits and determine their stability properties.

c. Sketch as much of the bifurcation diagram as you can for this case.

d. Explain what you expect to occur eventually in iterations of this mapping in each of the following two cases: (i) $a = 0.5, \ x_0 = 0.6$, and (i) $a = 1.64, \ x_0 = 0.4$.

5. The one-parameter family of three dimensional non-linear ordinary differential equations

$$\frac{dx}{dt} = 5y - 5x$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = xy - 3z$$

undergoes a sequence of bifurcations as the parameter r (> 0) increases in value.

a. Show that there is only one equilibrium point for $r < 1$ but that there are three equilibrium points for $r > 1$.

b. Show, further, that the $r < 1$ equilibrium point is asymptotically stable for $r < 1$ and unstable for $r > 1$.

c. Show that the eigenvalue equation for the Jacobian matrix at the additional equilibrium points is

$$\lambda^3 + 9\lambda^2 + 3(r+5)\lambda + 30(r-1) = 0$$

and, hence, deduce that the two additional equilibrium points are asymptotically stable only for $1 \lesssim r < 55$.

d. What special feature occurs at $r = 55$?