Ollscoil na hÉireann, Galway
National University of Ireland, Galway

GX 2155

Semester I Examinations, 2003/2004

Exam Code(s)	4BS4, 4BS9, 4BA4, 3BS5,3BS3,3BA1
Exam(s)	Fourth Science, Third Science
Module Code(s)	MP494
Module(s)	Partial Differential Equations
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Micheál Ó Confhaola
	Professor J.N. Flavin
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Instructions: Attempt **THREE** questions

Duration	**TWO HOURS**
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	2 (excluding this page)
Department(s)	Mathematical Physics

1. a. Consider the following system of three first order partial differential equations for functions $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} = 0,$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} = 0.$$

Write this system in the form

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y} = \mathbf{0}$$

where $\mathbf{u} = (u,v,w)^t$ and $\mathbf{A}$, $\mathbf{B}$ are 3×3 matrices. Hence show that the system is hyperbolic.

b. Consider the following partial differential equation for a function $u = u(x,y)$:

$$y^2\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x} + 6y.$$

Classify this equation according to type and reduce it canonical form. Hence find its general solution.

2. The thickness of paint on a vertical wall at height x and time t is given by $u = u(x,t)$. The evolution of this function is governed by the following partial differential equation:

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 0, \; t > 0, -\infty < x < \infty.$$

Show that the characteristics are straight lines and that the Rankine-Hugoniot condition on a shock $x = s(t)$ is (in the usual notation)

$$\frac{ds}{dt} = \frac{1}{3}\frac{[u^3]_-^+}{[u]_-^+}.$$

A stripe of paint is applied at $t = 0$ so that

$$u(x,0) = \begin{cases} 0 \text{ for } x < 0 \text{ or } x > 1, \\ 1 \text{ for } 0 < x < 1. \end{cases}$$

Show that for small enough t,

$$u(x,t) = \begin{cases} 0 \text{ for } x < 0, \\ (x/t)^{1/2} \text{ for } 0 < x < t, \\ 1 \text{ for } 0 < x < s(t), \\ 0 \text{ for } x > s(t), \end{cases}$$

where the shock is $x = s(t) = 1 + t/3$. Explain why this solution changes at $t = 3/2$ and show that thereafter $s(t) = (3/2)^{2/3}t^{1/3}$. Sketch the various solutions regions in the (x,t) plane. Sketch $u(x,t)$ as a function of x for various fixed times t.

3. Let D be a bounded, open plane domain, C being its boundary. Consider the problem:

$$u(x,y) \in C^2(D) \cap C(D \cup C)$$

and

$$\nabla^2 u + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} = -F(x,y) \text{ in } D,$$

where $F < 0$, where D, E are uniformly bounded, and where (x,y) denote rectangular Cartesian coordinates. Prove that the maximum of u occurs on the boundary C.
If the condition $F < 0$ in the foregoing is replaced by $F \leq 0$, deduce that if $u \leq M$ on C, then $u \leq M$ in D, M being a constant. [Hint: Consider $v = u + \varepsilon e^{\alpha x}$ where ε is a positive constant and α is a sufficiently large constant.]
Suppose that $F \equiv 0$ in D. Deduce that

$$u \geq m \text{ on } C$$

implies that

$$u \geq m \text{ in } D,$$

m being a constant.

4. a. Consider the eigenvalue problem

$$\phi(x,y) \in C(D \cup C) \cap C^2(D).$$
$$\nabla^2\phi + \lambda\sigma(x,y)\phi = 0 \text{ in } D,$$
$$\phi = 0 \text{ on } C,$$

where $\sigma(x,y) > 0$, and where D, C are as stated in the previous question. State a theorem giving a lower bound for the lowest eigenvalue in terms of a smooth, positive function. Use it to obtain such a lower bound when D is the ellipse

$$0 \leq x^2/a^2 + y^2/b^2 < 1,$$

a, b being positive constants, and

$$\sigma = \sigma_0(2a^2 - x^2)$$

where σ_0 is a positive constant.

b. Consider the initial boundary value problem

$$u(x,t) \in C^2[0 \leq x \leq \pi, t \geq 0]$$

satisfies

$$u_t + u_{xx} = 0,\ 0 < x < \pi,\ t > 0,$$
$$u(0,t) = u(\pi,t) = 0,\ t \geq 0,$$
$$u(x,0) = f(x).$$

Prove that this problem is ill-posed as follows: take

$$f(x) = \sin(nx)/n$$

n being a positive integer, and prove that the separable variable solution in this case $u_n(x,t)$, satisfies the following: given any positive constant ε,

$$|u_n(x,0)| < \varepsilon,\ |u_n(\pi/(2n),t)| > t/\varepsilon$$

provided $n > 1/\varepsilon$.