Ollscoil na hÉireann, Galway GX 2157
National University of Ireland, Galway

Spring Examinations, 2003/2004

Exam Code(s)	3BS3,3CS1,3CS3,3IF1,3PT1,4B3,4CS2
Exam(s)	Third and Fourth Science
Module Code(s)	CS305
Module(s)	Computing Techniques Of Applied Mathematics
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Micheál Ó Confhaola Dr. M.G. Meere

Instructions:	Attempt **THREE** questions
Duration	**TWO HOURS**
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	3 (excluding this page)
Department(s)	Mathematical Physics

1. Consider the following two point boundary value problem for a function $u = u(x)$:

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \text{ for } 0 < x < 1,$$

$$u = 0 \text{ on } x = 0 \text{ and } \frac{du}{dx} = 1 \text{ on } x = 1.$$

a. Show that the weak form of this boundary value problem is given by

$$B(u,w) = l(w)$$

where

$$B(u,w) = \int_0^1 \left\{ \frac{du}{dx}\frac{dw}{dx} - uw \right\} dx \text{ and } l(w) = -\int_0^1 x^2 w \, dx + w(1)$$

and where $w = w(x)$ is a sufficiently smooth weight function. In your derivation you should identify the primary variable and the secondary variable, as well as classifying the boundary conditions.

b. Seeking an approximate solution to this problem of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots. + c_N\phi_N(x)$$

where the $\{\phi_i(x)\}_{i=1}^N$ are pre-selected approximation functions, show that for the Rayleigh-Ritz procedure, the $\{c_i\}_{i=1}^N$ are determined by solving the linear equations

$$\sum_{j=1}^{N} B_{ij}c_j = F_i \text{ for } i = 1,2,\ldots,N$$

where

$$B_{ij} = B(\phi_i,\phi_j),\ F_i = l(\phi_i) - B(\phi_i,\phi_0).$$

c. Calculate the Rayleigh-Ritz approximation to the above problem for the case $N = 2$ with

$$\phi_0(x) = 0,\ \phi_1(x) = x,\ \phi_2(x) = x^2.$$

2. Consider the following two point boundary value problem for a function $u = u(x)$:

$$A(u) = f(x),\ 0 < x < 1,$$

$$\frac{du}{dx} = 0 \text{ on } x = 0 \text{ and } u = 0 \text{ on } x = 1,$$

where

$$A(u) = -\frac{d^2u}{dx^2} + 16u,\ f(x) = 4.$$

a. Seeking an approximate solution to this problem of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots. + c_N\phi_N(x)$$

where the $\{\phi_i(x)\}_{i=1}^N$ are pre-selected approximation functions, show that for the Galerkin method, the $\{c_i\}_{i=1}^N$ are determined by solving the linear equations

$$\sum_{j=1}^{N} A_{ij}c_j = F_i \text{ for } i = 1,2,\ldots,N$$

where

$$A_{ij} = \int_0^1 \phi_i A(\phi_j) dx,\ F_i = \int_0^1 \phi_i (f - A(\phi_0)) dx.$$

b. Calculate the Galerkin approximation to the above problem for the case $N = 2$ with

$$\phi_0(x) = 0,\ \phi_1(x) = \cos(\pi x/2),\ \phi_2(x) = \cos(3\pi x/2).$$

3. Consider the following two point boundary value problem for a function $u = u(x)$:

$$A(u) = f(x),\ 0 < x < 1,$$

$$u = 1 \text{ on } x = 0 \text{ and } \frac{du}{dx} = -1 \text{ on } x = 1,$$

where

$$A(u) = -\frac{d^2u}{dx^2} + xu,\ f(x) = 1 - x.$$

a. An approximate solution to this problem is sought of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots + c_N\phi_N(x)$$

where the $\{\phi_i(x)\}_{i=1}^{N}$ are pre-selected approximation functions. The residue, R, is defined by $R = A(U_N) - f$. Show that by minimizing $\int_0^1 R^2 dx$ with respect to the c_j's, $j = 1,2,\ldots,N$, that (the least-squared method)

$$\sum_{j=1}^{N} A_{ij}c_j = F_i \text{ for } i = 1,2,\ldots,N$$

where

$$A_{ij} = \int_0^1 A(\phi_i)A(\phi_j)dx,\ F_i = \int_0^1 A(\phi_i)(f - A(\phi_0))dx.$$

b. Calculate the least-squared approximation to the above problem for the case $N = 1$ with

$$\phi_0(x) = 1 - x,\ \phi_1(x) = x(x-2).$$

4. Consider the following two point boundary value problem for a function $u(x)$:

$$-\frac{d^2u}{dx^2} + u = 2 \text{ for } 0 < x < 1,$$

$$u = 0 \text{ on } x = 0 \text{ and } \frac{du}{dx} = 1 \text{ on } x = 1.$$

The interval $0 < x < 1$ is split into three equal subintervals of length 1/3 with a view to estimating the solution using the Finite Element Method.
The elements are denoted by $(x_{1,e}, x_{2,e})$ with $e = 1,2,3$. Estimating u in the interval $(x_{1,e}, x_{2,e})$ by the linear approximation

$$U_e = u_{1,e}\psi_{1,e}(x) + u_{2,e}\psi_{2,e}(x)$$

where $\psi_{1,e}(x) = 3(x_{2,e} - x)$, $\psi_{2,e}(x) = 3(x - x_{1,e})$, the element equations are given by (you are not required to show this) :

$$\sum_{j=1}^{2} K_{ij,e}u_{j,e} - f_{i,e} - (-1)^i Q_{i,e} = 0 \text{ for } i = 1,2 \text{ and } e = 1,2,3,$$

where

$$K_{ij,e} = \int_{x_{1,e}}^{x_{2,e}} \left(\frac{d\psi_{i,e}}{dx}\frac{d\psi_{j,e}}{dx} + \psi_{i,e}\psi_{j,e} \right) dx,\ f_{i,e} = \int_{x_{1,e}}^{x_{2,e}} 2\psi_{i,e}dx \text{ for } i,j = 1,2 \text{ and } e = 1,2,3,$$

$$Q_{1,e} = \left(\frac{du}{dx}\right)_{x=x_{1,e}},\ Q_{2,e} = \left(\frac{du}{dx}\right)_{x=x_{2,e}} \text{ for } e = 1,2,3.$$

a. Show that

$$\mathbf{k}^e = (K_{ij,e}) = \begin{pmatrix} 28/9 & -53/18 \\ -53/18 & 28/9 \end{pmatrix},\ \mathbf{f}^e = (f_{i,e}) = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} \text{ for } e = 1,2,3.$$

b. Writing $u_{2,1} = u_{1,2} = U_2$, $u_{2,2} = u_{1,3} = U_3$, $u_{2,3} = U_4$, show that (cont'd overleaf)

Q4. cont'd:

$$\begin{pmatrix} 56/9 & -53/18 & 0 \\ -53/18 & 56/9 & -53/18 \\ 0 & -53/18 & 56/9 \end{pmatrix} \begin{pmatrix} U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ 4/3 \end{pmatrix}.$$

5. a. Consider the following system of linear equations:

$$\begin{pmatrix} 1 & 4 & 4 \\ 0 & 2 & 3 \\ 1 & 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -8 \end{pmatrix}$$

i. Write a MAPLE command to define the coefficient matrix **A** (say) in the above system of equations.

ii. Write MAPLE commands to calculate $\mathbf{A}^{-1}$, $\mathbf{A}^{t}$, $\mathbf{A}^{15}$ and the determinant of **A**.

iii. Write MAPLE commands to calculate the eigenvalues and eigenvectors of **A**.

iv. Write MAPLE commands to calculate the characteristic polynomial of **A**, and to verify that **A** satisfies this characteristic polynomial.

v. Write MAPLE commands to solve the linear system of equations defined above.

b. Write MAPLE commands to solve each of the following problems:

i. $\frac{dy}{dt} = y^5,\ y(0) = 1,$

ii. $-\frac{d^4y}{dx^4} + y = x,\ y(0) = 1,\ \frac{dy}{dx}(0) = 0, y(1) = 0,\ \frac{dy}{dx}(1) = 1.$