Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

GX 2166

Semester II Examinations, 2003/2004

Exam Code(s)	2BA1, 2BS1, 2EL1, 2PT1
Exam(s)	**Second Science**
Module Code(s)	MP204
Module(s)	Mathematical Physics(Honours)
Paper No.	SUMMER
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan;
Internal Examiner(s)	Dr. M. S. Ó Confhaola;
	Dr. B. Gleeson.
Instructions:	Attempt *FIVE* questions, at least *TWO* from each section.
Duration	*THREE* hours
No. of Answer books	____________
Requirements:	____________
Handout	____________
MCQ	____________
Statistical Tables	YES, LOG TABLES
Graph Paper	____________
Log Graph Paper	____________
Other Material	____________
No. of Pages	4 PAGES (Excluding Front Page)
Department(s)	MATHEMATICAL PHYSICS

Section A

1. Two wheels, each of mass m and radius a, are released simultaneously from rest at the top of an inclined plane. The coefficient of friction is

$$\mu = \tfrac{3}{5} \tan \alpha$$

where α is the angle of inclination of the plane. The first wheel is a uniform solid disk. The second wheel is also a uniform disk, but has had a circular hole cut out of its centre so that its axial moment of inertia is

$$\mathrm{I} = \tfrac{1}{3} m a^2 .$$

If the wheels remain vertical throughout, determine in each case, whether the wheel is rolling or sliding. Find the linear acceleration of each wheel down the incline. Determine which wheel reaches the bottom first, assuming that they move along parallel straight paths down the incline.

2. A light string passes over a fixed smooth pulley. It carries a mass 7 M at one end, the other being attached to a smooth pulley of mass 3 M over which passes a second light string whose ends carry masses 2 M and 3 M. Obtain Lagrange's equations of motion for the system if you are told that the pulleys are small and their rotational motion can safely be ignored. Use these equations to find the accelerations of each of the four masses.

3. A thin circular hoop of radius a and mass m is allowed to oscillate in its own plane with one point O of the hoop fixed in position. A bead of mass $3m$ is threaded on the hoop and is constrained to move in a frictionless manner around the hoop while the hoop is oscillating. You may assume that

$$\mathrm{T} = \tfrac{1}{2} m a^2 \left[5\dot{\theta}^2 + 3\dot{\phi}^2 + 6\dot{\theta}\dot{\phi}\right],$$
$$\mathrm{V} = \tfrac{1}{2} m a g \left[2\theta^2 + \tfrac{3}{2}\phi^2\right],$$

where θ is the angle that the line joining the centre of the hoop to the fixed point makes with the downward vertical and ϕ is the angle that the line joining the centre of the hoop to the bead makes with the downward vertical.

By finding the matrices T_{ij} and V_{ij}, corresponding to the kinetic and potential energies T and V respectively, show that the natural frequencies for small oscillations about the configuration of stable equilibrium are

$$\sqrt{\frac{g}{4a}} \quad \text{AND} \quad \sqrt{\frac{2g}{a}}$$

and hence show that $4\theta + 3\phi$ and $\phi - \theta$ are suitable natural coordinates.

4. (a) Given the definition of the Hamiltonian

$$\mathrm{H}(q_i, p_i, t) = p_i \dot{q}_i - \mathrm{L}(q_i, \dot{q}_i, t),$$

where the generalised momentum p_i is defined by

$$p_i = \frac{\partial \mathrm{L}}{\partial \dot{q}_i},$$

derive Hamilton's Equations, from Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathrm{L}}{\partial q_i},$$

where $i = 1, 2, 3, ..., \mathrm{N}$.

(b) A light string passes over a fixed smooth pulley. It carries a mass $6M$ at one end, the other being attached to a smooth pulley of mass $3M$ over which passes a second light string whose ends carry masses $2M$ and M.

If you are told that the pulleys are small and that their rotational motion can safely be ignored, show that the Lagrangian for the system is given by

$$\mathrm{L} = 6M\dot{x}^2 + \tfrac{3}{2}M\dot{y}^2 + M\dot{x}\dot{y} - Mgy,$$

choosing as generalised coordinates x, y, the displacement of the pulley, and the $2M$ mass (relative to the pulley), respectively.

Find the Hamiltonian for this system, and hence write down the Hamilton Equations of motion.

5. (a) Explain and prove the Relativistic length contraction result $\overline{\mathrm{L}} > \mathrm{L}$.

(b) Explain and prove the Relativistic time dilation result $\mathrm{T} > \overline{\mathrm{T}}$.

(c) Show that if a particle moves along the x axis of frame S with speed u then it moves along the $\bar{x}$ axis of $\overline{\mathrm{S}}$ with speed $\bar{u}$ where

$$\bar{u} = \frac{u - v}{1 - uv/c^2},$$

where as usual, v is the speed of frame $\overline{\mathrm{S}}$ along the x axis.

(d) Three particles are moving along the positive x axis, in the positive direction. The first particle has a speed of $0.6c$ relative to the second, the second has a speed of $0.8c$ relative to the third and the third has a speed of $0.5c$ relative to the laboratory. What is the speed of the first particle relative to the laboratory?

Section B

6. Solve the following partial differential equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} : \quad 0 < x < L, \quad t > 0,$$

where α is a real positive constant, subject to both the conditions

$$u(0,t) = u(L,t) = 0, \quad t \geqslant 0,$$

on the boundary, and the initial condition

$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant L,$$

using the method of separation of variables.

7. Solve the following partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 : \quad 0 < \theta < \alpha, \quad a < r < b,$$

where $\alpha < 2\pi$, subject to the conditions

$$u_\theta(r,0) = u_\theta(r,\alpha) = 0, \quad a < r < b,$$

on the straight edges, and the condition

$$u(a,\theta) = 0 \quad u(b,\theta) = f(\theta), \quad 0 < \theta < \alpha,$$

on the curved edge, where $a, b > 0$, using the method of separated variables.

8. (a) Using the substitution $u = x - \mathrm{c}\,t$, $v = x + \mathrm{c}\,t$, or otherwise, prove that the most general solution of the Wave Equation

$$\frac{\partial^2 \phi}{\partial t^2} = \mathrm{c}^2 \frac{\partial^2 \phi}{\partial x^2}$$

is $\phi(x,t) = f(x - \mathrm{c}\,t) + g(x + \mathrm{c}\,t)$, where $\mathrm{c}\ (> 0)$ is a constant.

(b) Demonstrate this by substituting the solution

$$\phi(x,t) = \mathrm{A}\sin(x - \mathrm{c}\,t) + \mathrm{B}\cos(x + \mathrm{c}\,t)$$

into the above Wave Equation.

9. (a) Find the general solution to Bessel's equation of order $\frac{1}{4}$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\tfrac{1}{16}\right)y\left(x\right)=0$$

using the Method of Frobenius.

(b) Use the transformation $y\left(x\right)=xu\left(x\right)$ in the ordinary differential equation

$$x^2y''-2xy'+\left(2-k^2x^2\right)y=0,$$

where $k>0$, and hence write down the general solution for $y\left(x\right)$.

10. (a) Use the following cross-sectional measure of $u\left(x,t\right)$:

$$F\left(t\right)=\int_0^L u^2\left(x,t\right)dx,$$

where $u\left(x,t\right)$ is the solution of the following equation

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}:\quad 0<x<L,\quad t>0,$$

in the usual notation, subject to the boundary conditions

$$u\left(0,t\right)=0,\quad t\geqslant 0,$$
$$u\left(L,t\right)=0,\quad t\geqslant 0,$$

and subject to the initial condition

$$u\left(x,0\right)=f\left(x\right),\quad 0\leqslant x\leqslant L,$$

to determine whether the above Initial Boundary Value Problem, is unique. (Assume that the solution $u\left(x,t\right)$ exists).

(b) Consider the BVP corresponding to Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0:\quad 0<x<L,\quad 0<y<H,$$

subject to the boundary conditions

$$u_y\left(0,y\right)=u\left(L,y\right)=0,\quad 0<y<H,.$$

Establish a '*conservation law*' for this Boundary Value Problem.