Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

GX 2169

Semester II Examinations, 2003/2004

Exam Code(s)	2BS1, 2CS1, 2EL1, 2PT1, 3CS1.
Exam(s)	**Second Science and Arts**
Module Code(s)	MP208
Module(s)	Methods of Mathematical Physics(Honours)
Paper No.	PAPER 2
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan;
Internal Examiner(s)	Dr. M. S. Ó Confhaola;
	Dr. B. Gleeson;
	Dr. F. Ó Dea.
Instructions:	Attempt *THREE* questions, at least *ONE* from each section.
Duration	*TWO* hours
No. of Answer books	
Requirements:	
Handout	
MCQ	
Statistical Tables	YES, LOG TABLES
Graph Paper	
Log Graph Paper	
Other Material	
No. of Pages	3 PAGES (Excluding Cover Page)
Department(s)	MATHEMATICAL PHYSICS

Section A

1. (a) Find the derivative, with respect to distance, of the function

$$\phi(x, y, z) = 3x^2z + 3y - z^3,$$

along the parametric curve

$$\mathbf{r}(u) = (u - 1, u^2 - 1, 2 - 2u),$$

at the origin $(0, 0, 0)$.

(b) Prove that the point $(2, -1, 2)$ lies on each of the surfaces

$$x^2 - 2y^2 + 2z^2 = 10 \quad \text{AND} \quad z = x^2 + y^2 - 3.$$

Prove that these surfaces intersect at right angles at this point.

(c) Verify the vector identity

$$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \operatorname{grad}(\operatorname{div} \mathbf{F}) - \nabla^2 \mathbf{F}$$

for the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$.

2. (a) Consider the integral

$$\int_{-3}^{3} \int_{x^2}^{9} dy dx.$$

Sketch the region of integration A. Change the order of integration and hence evaluate the area A.

(b) Use plane polar coordinates to evaluate

$$\iint_{\mathrm{A}} 2(x^2 - y^2)\, dx dy$$

where A is the portion of the disc $x^2 + y^2 \leqslant 4$, $x, y \geqslant 0$.

(c) Evaluate the double integral

$$\iint_{\mathrm{A}} (x + y)\, d\,\mathrm{A},$$

where A is the region bounded by the lines $x + y = 1$, $x + y = 3$, $2x - y = 0$, $2x - y = 4$ by making the change of variable $u = x + y$, $v = 2x - y$.

3. Evaluate the line integral

$$\int_{(0,0,0)}^{(1,2,-3)} \mathbf{F} \cdot \mathbf{dr},$$

for the vector field

$$\mathbf{F} = (2xy^2 - yz)\,\mathbf{i} + (2x^2y - xz)\,\mathbf{j} - xy\mathbf{k}$$

along each of the following paths:

(a) the straight line joining $(0,0,0)$ directly to $(1,2,-3)$;

(b) the parametric curve $\mathbf{r}\,(u) = (u, 2u^2, -3u^3)$

(c) the successive straight line segments from $(0,0,0)$ to $(0,2,0)$, from $(0,2,0)$ to $(0,2,-3)$ and from $(0,2,-3)$ to $(1,2,-3)$, in this order.

Prove that the vector field $\mathbf{F}$ is conservative. Find, by line integration or otherwise, the corresponding scalar potential $\phi\,(x,y,z)$ for this field $\mathbf{F}$.

Section B

4. The divergence theorem of Gauss can be written in the form

$$\iiint_{\mathrm{V}} \boldsymbol{\nabla} \cdot \mathbf{A}\, d\,\mathrm{V} = \iint_{\mathrm{S}} \mathbf{A} \cdot \hat{\mathbf{n}}\, d\,\mathrm{S}\,.$$

Explain what is meant by V, S and $\hat{\mathbf{n}}$ in this equation. Verify the divergence theorem for the vector field

$$\mathbf{A} = x^2y\mathbf{i} - xy^2\mathbf{j} - 2xz^2\mathbf{k},$$

for the cylindrical region $x^2 + y^2 \leqslant 16$, $0 \leqslant z \leqslant 3$.

5. Spherical Polar co-ordinates (r, θ, ϕ) are related rectangular Cartesian co-ordinates (x, y, z) by the relationships:

$$x = r\sin(\theta)\cos(\phi)\,, \quad y = r\sin(\theta)\sin(\phi)\,, \quad z = r\cos(\theta)\,.$$

(a) Calculate the scale factors h_r, h_θ and h_ϕ and verify that the system is orthogonal.

(b) Find $\boldsymbol{\nabla}\Psi$ in terms of this co-ordinates system.

(c) Find $\nabla^2\Psi$ in terms of these co-ordinates.
[You may assume that, in general, for an orthogonal system

$$\nabla^2\Psi = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Psi}{\partial u_3}\right)\right\}\Bigg].$$

6. (a) Define Cartesian tensors of ranks one and two.

   (b) Determine the rotation matrix associated with a rotation of Cartesian frame about its x_3 axis, the rotation being of angle α in an anti-clockwise direction. If the new frame is rotated by an angle β about the new x_2 axis, calculate the rotation matrix for the composite rotation.

   (c) A tensor of rank two has components

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   with respect to a given Cartesian frame. The frame is rotated by an angle α about its x_3 axis (see part (*b*) of this question). Find the components of the tensor in the rotated frame.