Ollscoil na hÉireann, Gaillimh GX 2172 - GO
National University of Ireland, Galway

Semester II Examinations, 2003/2004

Exam Code(s)	2BE1, 2BG1, 2BI1, 2BI2, 2BM1, 2BN1, 2BP1, 2BV1
Exam(s)	Second Engineering
Module Code(s)	MP250
Module(s)	Mathematical Physics
Paper No	2
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan
Internal Examiner(s)	Dr. M. Ó Confhaola
	Dr. T. N. Sherry
	Dr. M. Meere

Instructions:

Answer: THREE questions from Section A and
TWO questions from Section B

All questions carry the same marks.

A compendium of useful formulae is attached.

Duration	3 hours
No. of Answer books	1
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	5
Department(s)	Mathematical Physics

SECTION A

1. The volume of water that flows over a spillway per second per unit length along the crest of the spillway, Q, depends on the height h of the water surface above the crest of the spillway, the acceleration due to gravity g, a length l that specifies the size of the cross-section of the spillway, the mass density ρ of the water, the dynamic viscosity μ of the water and the roughness height e of the concrete.

   Show that Q does not depend on μ and ρ separately but only on the ratio μ/ρ. What is the general form of a dimensionally homogenous equation for Q in terms of the other variables?

   You are told that Q depends linearly on g. How then does Q depend on ρ and on μ? Can you infer the dependence of Q on h? Explain your answer.

2. A compound pendulum is formed out of a uniform spherical ball, of mass $5M$ and radius a, and a thin uniform rod of mass M and length $6a$. The rod is rigidly attached to the surface of the ball, pointing radially outwards from the centre of the ball. The other end of the rod is fixed in position and the pendulum is allowed to oscillate freely in the vertical plane of the rod.

   a. Deduce the equation of motion of the compound pendulum from the principle of angular momentum and, hence, identify the length of the equivalent simple pendulum.

   b. The pendulum is initially held with the rod at an angle $\pi/4$ to the downward vertical. What angular speed must the pendulum be given to ensure that it completes at least one complete revolution?

3. A wheel consists of a thin rim of mass M and radius a and 12 equally spaced spokes each of mass m, which may be considered as thin rods of length a terminating at the centre of the wheel. Show that the axial moment of inertia of the wheel is given by

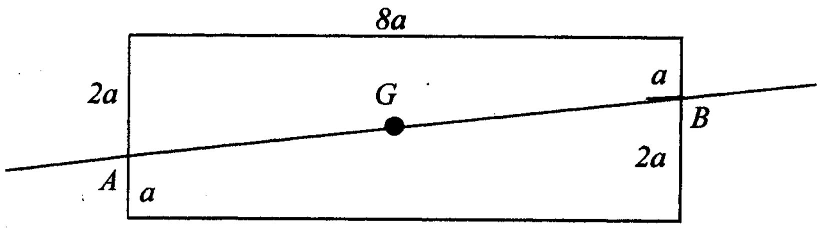
$$I = (M + 4m)a^2 .$$

   The wheel is set in motion along a straight horizontal path on a rough plane (coefficient of friction μ) with zero initial linear velocity and initial spin ω. Show that, if the plane of the wheel remains vertical throughout the motion, the wheel will slide for a time

$$\frac{M+4m}{2M+16m}\,\frac{a\omega}{\mu g}$$

   and that, after this time, it will roll. Determine in which direction the wheel is moving when it rolls.

4. A uniform rectangular plate, with mass m and sides of length $3a$ and $8a$, is constrained to rotate about the fixed axis AB, as indicated in the diagram below, with constant angular speed ω.



a. Express the angular momentum of the plate relative to its centre of mass G in terms of ω, m and a.

b. Find the torque of the reaction couple which must act on the plate to maintain the motion.

c. Find, also, a realisation of this reaction couple in terms of reaction forces acting on the plate.

5. A particle of mass m is free to slide on a smooth straight horizontal wire. A second particle of mass $7m$ is suspended from the first particle by a heavy rigid rod, of length $2l$ and mass $6m$. The system is allowed to swing in the vertical plane of the rod and wire.
Choosing as coordinates the distance s of the mass m from a fixed point along the wire and the angle θ which the rod makes with the downward vertical, show that the Lagrangian function for this system is

$$7\,m\dot{s}^2 + 20\,ml\dot{s}\dot{\theta}\cos\theta + 18\,ml^2\dot{\theta}^2 + 20mgl\cos\theta$$

and, hence, obtain Lagrange's equations of motion for the system.

SECTION B

6. a. The curve C is described by the parametric equation

$$\mathbf{r} = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t^2\mathbf{k}, \quad -\infty < t < +\infty$$

and passes through the point $(0, 1, \pi^2/4)$. Find the unit tangent vector to the curve C at this point. Hence, or otherwise, find the directional derivative in the direction of the curve of the scalar field

$$\phi = y^2z^2 + x^2z^2 + x^2y^2$$

at the point $(0, 1, \pi^2/4)$.

b. Verify, by calculating each term separately, the identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

for the vector fields $\mathbf{A} = 2xy\mathbf{i} - 3x^2\mathbf{j} + z^2\mathbf{k}$ and $\mathbf{B} = x\mathbf{i} + y^2\mathbf{j} + z^3\mathbf{k}$.

7. Evaluate the line integral

$$\int_{(0,0,0)}^{(1,-1,-1)} \mathbf{F}.d\mathbf{r}$$

for the vector field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

along the following paths:

a. the curve $x = t, y = -t^2, z = -t^3$ from $t = 0$ to $t = 1$.

b. the successive straight line segments $(0,0,0)$ to $(1,0,0)$ to $(1,-1,0)$ to $(1,-1,-1)$.

c. the straight line joining $(0,0,0)$ to $(1,-1,-1)$.

Are the results of these calculations sufficient to determine whether the field $\mathbf{F}$ is conservative or not?

8. Verify the divergence theorem for the vector field

$$\mathbf{F} = \frac{1}{2}(x^2 - y^2)\,\mathbf{i} + \frac{1}{2}(y^2 - z^2)\,\mathbf{j} + \frac{1}{2}(z^2 - x^2)\,\mathbf{k}$$

when the volume of integration is the cube bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x = 1$, $y = 1$, $z = 1$.

USEFUL FORMULAE

Dimensional Analysis

The coefficient of viscosity μ is defined by a dimensionally homogeneous equation of the form

$$\text{force per unit area} = \mu \cdot \big(\text{gradient of a velocity}\big)$$

Rigid Body Dynamics (for a rigid body free to move with one point fixed)

Principle of Angular Momentum:

$$\left(\frac{d}{dt}\mathbf{h}_O\right)' + \boldsymbol{\omega} \times \mathbf{h}_O = \mathbf{C}_O$$

Principal Moments of Inertia of a uniform rectangular plate:

If the edges of the plate are $2a$ and $2b$ and the mass of the plate is m, then the principal moments of inertia at the centre of mass of the plate are $\frac{1}{3}mb^2$, $\frac{1}{3}ma^2$ and $\frac{1}{3}m(a^2+b^2)$, the corresponding principal axes being parallel to the sides of length $2a$ and $2b$, in turn, and perpendicular to the plate.

Lagrange's Equations for a Simple Lagrangian System

For a system with n degrees of freedom and corresponding coordinates

$$q_i, \; i = 1,2,\ldots,n,$$

the Lagrangian function is given by

$$L(q,\dot{q}) = T(q,\dot{q}) - V(q).$$

The equations of motion are then given by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = 0, \quad i = 1,2,\ldots,n.$$

Vector Differential Calculus

The vector differential operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

The divergence theorem of Gauss

Given a volume V bounded by a closed surface S, a differentiable vector field $\mathbf{F}$ defined throughout V and on S and a unit *outward* normal vector $\mathbf{n}$ on S, then

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS\,.$$