Ollscoil na hÉireann, Gaillimh GX 2173
National University of Ireland, Galway

Semester II Examinations, 2003/2004

Exam Code(s)	3BS3,3BS9
Exam(s)	Third Science, Fourth Science
Module Code(s)	MP303
Module(s)	Quantum Mechanics (Pass)
Paper No	1
Repeat Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Mícheál Ó Confhaola
	Dr. Michael Tuite

Instructions: Full marks for **FIVE** correctly answered questions.

Duration	3hrs
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	4
Department(s)	Mathematical Physics

1. a. Describe the probabilistic interpretation of a wave function $\Psi(x,t)$ for a 1dimensional particle. How is the expectation of an observable $Q(x,\hat{p})$ defined?
   b. The probability current is defined to be $J(x,t) = \frac{i\hbar}{2m}(-\frac{\partial\psi^*}{\partial x}\psi + \psi^*\frac{\partial\psi}{\partial x})$. Show that
$$\frac{d}{dt}\text{prob}(a \le x \le b) = J(b,t) - J(a,t).$$
   c. Let $\{\psi_n(x), n = 1,2,..\}$ be an orthonormal set of energy eigenfunctions with eigenvalues E_n. Let $\Psi(x,t)$ be a general wave function with initial form $\Psi(x,0) = \sum_{n\ge1} c_n\psi_n(x)$. Show that
$$\Psi(x,t) = \sum_{n\ge1} c_n\psi_n(x)\exp(-\frac{iE_nt}{\hbar}).$$
   d. Hence show that
$$\langle\hat{H}\rangle = \sum_{n\ge1} E_n|c_n|^2.$$

2. Consider a particle of mass m moving along a line with position x for $0 \le x \le \pi$ in an infinite square well with potential
$$V(x) = \begin{cases} 0, & 0 \le x \le \pi \\ \infty, & \text{otherwise} \end{cases}$$
   a. Show that the normalized energy eigenfunctions are
$$\psi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx),$$
   with energy $E_n = n^2\hbar^2/2m$.
   b. Find the uncertainties σ_x, σ_p for the ground state $\psi_1(x)$.
   c. Show that the Heisenberg uncertainty principle $\sigma_x\sigma_p \ge \frac{\hbar}{2}$ is satisfied.

3. Consider the 1-dimensional Harmonic Oscillator system of mass m with Hamiltonian
$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega x^2}{2}.$$
Define the ladder operators
$$a = \frac{1}{\sqrt{2m}}(i\hat{p} + m\omega x), a^\dagger = \frac{1}{\sqrt{2m}}(-i\hat{p} + m\omega x).$$
   a. Show that $aa^\dagger = \hat{H} + \frac{\hbar\omega}{2}$ and $a^\dagger a = \hat{H} - \frac{\hbar\omega}{2}$.
   b. Show that with $\xi = x\sqrt{m\omega/\hbar}$ then
$$a = \sqrt{\frac{\hbar\omega}{2}}(\frac{d}{d\xi} + \xi), \quad a^\dagger = \sqrt{\frac{\hbar\omega}{2}}(-\frac{d}{d\xi} + \xi).$$
   c. Show that if ψ is an energy eigenfunction with energy eigenvalue E then $a^\dagger\psi$ and $a\psi$ are eigenfunctions with energy $E + \hbar\omega$ and $E - \hbar\omega$ respectively. Hence show that the energy eigenvalues are $E_n = (n + \frac{1}{2})\hbar\omega$ for $n = 0,1,2\ldots$
   d. Show that the ground state energy is $E_0 = \frac{1}{2}\hbar\omega$ with normalized eigenstate
$$\psi_0 = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-\frac{1}{2}\xi^2).$$

4.

a. Show that the energy eigenstates for a 1-dimensional system with even potential function $V(x)$ can be chosen to be either even or odd functions in x.

b. Consider the energy eigenstates for a particle of mass m moving in the potential well

$$V(x) = \begin{cases} -\frac{\hbar^2}{m} < 0, |x| < \frac{\pi}{4}, \\ 0, |x| > \frac{\pi}{4}. \end{cases}$$

Show that this system has a ground state with energy $E = -\frac{\hbar^2}{2m}$.

5.

a. Show for a free particle moving along a line that the energy eigenvalues are $E_k = \hbar^2 k^2/2m$ for any real k for eigenfunction

$$\psi_k(x) = \exp(ikx).$$

Show that $\psi_k(x)$ is an eigenfunction of the momentum operator $\hat{p}$ and explain why such momentum eigenstates are not physically possible.

b. A free particle (at some time t) is described by a Gaussian wave packet

$$\psi(x) = \frac{1}{(2\pi a^2)^{1/4}} \exp\left(-\frac{x^2}{4a^2}\right) \exp(ik_0 x).$$

for some constants $a > 0$ and k_0. Show that the uncertainty in position $\sigma_x = a$.

c. Show that the momentum space wave function $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) \exp(-ikx)dx$ is given by

$$\phi(k) = \frac{1}{(2\pi b^2)^{1/4}} \exp\left(-\frac{(k - k_0)^2}{4b^2}\right),$$

for $b = 2/a$. How are the uncertainties σ_x and σ_p related in this case?

6.

a. Prove the Schwarz inequality for a states f, g in a Hilbert space

$$|\langle f, g \rangle| \le \|f\| \cdot \|g\|,$$

where $\|f\| = \langle f, f \rangle$ etc.

b. Show for a given wavefunction ψ that $\sigma_x^2 = \|f\|^2$ for $f = (x - \langle x \rangle)\psi$ and $\sigma_p^2 = \|g\|^2$ for $g = (\hat{p} - \langle \hat{p} \rangle)\psi$

c. Hence prove the Uncertainty Principle for x and $\hat{p}$

7. Consider a particle of mass m moving in an infinite spherical potential well with potential

$$V(r) = \begin{cases} 0, & 0 \le r < a \\ \infty, & r > a \end{cases}$$

a. Show that the s-wave energy eigenfunctions are given by

$$\psi_n(r) = \frac{1}{\sqrt{2\pi a}} \frac{1}{r} \sin\left(\frac{n\pi r}{a}\right).$$

b. Find the energy eigenvalues.

c. Show that the probability of finding m in the region $r < a/2$ is one half.

You may assume that the Laplacian in spherical polar coordinates is given by
$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}.$

8. Consider the angular momentum operators L_1, L_2, L_3 obeying the commutation relations

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k.$$

Define the ladder operators $L_\pm = L_1 \pm iL_2$.

a. Show that the following commutation relations hold

$$[L^2, L_3] = 0,$$
$$[L_3, L_\pm] = \pm\hbar L_\pm.$$

b. Show that $L^2 = L_\pm L_\mp + L_3^2 \mp \hbar L_3$.

c. Hence show that the L^2 and L_3 eigenstates $|lm\rangle$ obey

$$L^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle,$$
$$L_3|lm\rangle = \hbar m|lm\rangle,$$

for $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ with $m = -l, -l+1, \ldots, l-1, l$.

d. Comment on the physical realization of such eigenstates for integer and half-integer l.

9. A two particle system consists of spin $s_1 = \frac{3}{2}$ particle together with a spin $s_2 = \frac{1}{2}$ particle. Describe the total spin eigenstates $|sm\rangle$ of the system in terms of the basis $\{\ |s_1m_1\rangle\ |s_2m_2\rangle\}$ and hence find the Clebsch-Gordan coefficients $C^{s_1s_2s}_{m_1m_2m}$.

You may assume the relation $L_\pm|lm\rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\,|lm \pm 1\rangle$ for orthonormal simultaneous eigenstates of L^2 and L_3.

Useful formulae

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}, \text{ 1-d Momentum Operator}$$

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(x), \text{ 1-d Hamiltonian Operator}$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t)$$
$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t), \text{ 1-d Schrödinger Equation}$$