Ollscoil na hÉireann, Gaillimh GX 2174
National University of Ireland, Galway

Semester II Examinations, 2003/2004

Exam Code(s)	3BS3,3BS5,3EL1,3EL2,3PT1,3PT2
	4BS2,4BS3,4BS4,4BS9,4CS2,1MF2,1SD1
Exam(s)	Third Science, Fourth Science, M.Sc.
Module Code(s)	MP307
Module(s)	Modelling II
Paper No	1
Repeat Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Mícheál Ó Confhaola
	Dr. Michael Tuite

Instructions: Full marks for **THREE** correctly anwered questions.

Duration	2hrs
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	3
Department(s)	Mathematical Physics

1.

(a) Describe what is meant by a finite ergodic Markov process with transition probability matrix P. Define the equilibrium probabilities π_i for a finite ergodic system and show that they obey the matrix equation $\Pi = \Pi P$ where Π is a row matrix with elements π_i. Describe a sufficient condition for P which guarantees ergodicity.

(b) Describe an infinite queue with a nearest-neighbour Markov process where the arrival and servicing patterns are independent of the queue size. Find under what conditions this system is ergodic and find the equilibrium probabilities.

(c) Customer arrivals in a queue are described by a Poisson process with parameter α. Show that the probability for k customers to arrive in time t is given by the Poisson distribution with parameter αt. Hence find the average number of arrivals per unit time.

(d) On average, a set of traffic lights lets through three cars every minute with 90 cars arriving every hour. Using a Poisson model for the arrival and servicing mechanism, find the equilibrium probability that more than 2 cars are queuing. What is the average size of the queue? How would the queue change if 180 cars arrived per minute?

2. A queue of maximum size r has a Poisson arrival pattern with parameter α_k and Poisson servicing pattern with parameter β_k where k is the queue size with $0 \le k \le r$.

(a) Explain why $\alpha_r = \beta_0 = 0$. Assuming the system is ergodic show that the equilibrium probabilities are for $0 < k \le r$ given by

$$\pi_k = \rho_k \rho_{k-1} \ldots \rho_1 \pi_0$$

where $\rho_k = \alpha_{k-1}/\beta_k$.

(b) Consider a telephone system exchange with r operators. Assuming the probability of a call arriving is independent of the queue size, find the equilibrium probabilitities.

(c) An exchange consists of two operators each of which can deal with 1 customer enquiry per minute on average. If 120 calls per hour arrive on average what is the equilibrium probability for all lines to be busy. What would be the effect of introducing a third operator?

3.

(a) Consider the notorious IBM RANDU multiplicative congruential random number generator

$$I_{k+1} = aI_k \bmod m, k = 0, 1, ..$$

with $m = 2^{31}$ for $a = 2^{16} + 3$ (which has period of 2^{16}). Show that all points (x_k, x_{k+1}, x_{k+2}) where $x_k = I_k/m$ must lie on one of 15 parallel planes within the unit cube.

(b) Assuming that pseudo-random numbers can be uniformly generated on [0, 1], describe algorithms which simulate

(i) an exponential decay process with probability density function $f(t) = \lambda \exp(-\lambda t)$ where $\lambda > 0$,

(ii) a normal distribution with mean 2 and standard deviation 1.

(c) Consider a finite ergodic Markov process with transition matrix p_{ij} and equilibrium probabilities π_i. Show that if the Detailed Balance condition

$$a_i p_{ij} = a_j p_{ji},$$

(i,j not summed) is satisfied by some numbers a_i, then $a_i = A\pi_i$ for an appropriate constant A.

(d) Hence describe the Metropolis algorithm as a Markov process which can simulate any given finite probability distribution π_i. Assuming that the algorithm is ergodic, show that the Detailed Balance condition $\pi_i p_{ij} = \pi_j p_{ji}$ is satisfied and hence the equilibrium probability distribution is π_i.

4. Consider a species in which no individuals live beyond 3 years. Divide the population into four age groups labelled by $n = 0, 1, 2$ and 3. Assume that only the second and third groups can reproduce. If for each group n, b_n denotes the birth rate and d_n the corresponding death rate, find a matrix $\mathbf{A}$ such that $\mathbf{P}(t+1) = \mathbf{A}\mathbf{P}(t)$ with

$$\mathbf{P} = \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

where $P_n(t)$ denotes the population size of age group n at time t.

(a) Show that the eigenvalues of $\mathbf{A}$ satisfy the equation

$$\lambda^4 - b_1(1-d_0)\lambda^2 - b_2(1-d_0)(1-d_1)\lambda = 0.$$

(b) If $d_0 = 0.2$, $d_1 = 0.2$, $d_2 = 0.4$, and $b_1 = 0.5$, $b_2 = 1.0$, show that there is one eigenvalue greater than 1. What is the significance of this?

(c) Considering the evolution of the species over three years, where the initial population is $P_0 = P_1 = P_2 = 10$ and $P_3 = 0$.

5. Consider an ecological system with a prey population of size $x(t)$ and a predator population of size $y(t)$ which satisfy the differential equations

$$\frac{dx}{dt} = x(2-y),$$

$$\frac{dy}{dt} = 8y(-1+x).$$

(a) Give an interpretation for the various terms in these equations.

(b) For small deviations away from the equilibrium point $(x,y) = (1,2)$, show that the system is periodic with period $\pi/2$.

(c) Show that in general the solution satisfies

$$x^4 y = Ke^{4x+y/2}$$

for some constant K. Sketch a plot of x vs. y and comment on the typical behaviour of this system in an application of this model.