Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

GX 2176

Semester II Examinations, 2003/2004

Exam Code(s)	3BS3,3BS5,3CS2,3PT1,3PT2,4BS4
Exam(s)	Third Science, Fourth Science
Module Code(s)	MP324
Module(s)	Quantum Mechanics (Honours)
Paper No	1
Repeat Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Mícheál Ó Confhaola
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Instructions: Full marks for **FIVE** correctly answered questions.

Duration	3hrs
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	5
Department(s)	Mathematical Physics

1. Consider a particle of mass m moving along a line with position x for $0 \leq x \leq \pi$ in an infinite square well with potential

$$V(x) = \begin{cases} 0, & 0 \leq x \leq \pi \\ \infty, & \text{otherwise} \end{cases}$$

a. Show that the normalized energy eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx),$$

with energy $E_n = n^2\hbar^2/2m$.

b. Find the uncertainties σ_x, σ_p for the nth state $\psi_n(x)$.

c. Show that the Heisenberg uncertainty principle $\sigma_x\sigma_p \geq \frac{\hbar}{2}$ is satisfied. Which state(s) come closest to satisfying the lower bound?

d. Suppose that at time $t = 0$ the initial state of the system is

$$\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x)).$$

What is the wave function $\Psi(x,t)$ for later t?

2. Consider the 1-dimensional Harmonic Oscillator system of mass m with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega x^2}{2}.$$

Define the ladder operators

$$a = \frac{1}{\sqrt{2m}}(i\hat{p} + m\omega x), a^\dagger = \frac{1}{\sqrt{2m}}(-i\hat{p} + m\omega x).$$

a. Show that $aa^\dagger = \hat{H} + \frac{\hbar\omega}{2}$ and $a^\dagger a = \hat{H} - \frac{\hbar\omega}{2}$.

b. Show that with $\xi = x\sqrt{m\omega/\hbar}$ then

$$a = \sqrt{\frac{\hbar\omega}{2}}\left(\frac{d}{d\xi} + \xi\right), \quad a^\dagger = \sqrt{\frac{\hbar\omega}{2}}\left(-\frac{d}{d\xi} + \xi\right).$$

c. Show that if ψ is an energy eigenfunction with energy eigenvalue E then $a^\dagger\psi$ and $a\psi$ are eigenfunctions with energy $E + \hbar\omega$ and $E - \hbar\omega$ respectively. Hence show that the energy eigenvalues are $E_n = (n + \frac{1}{2})\hbar\omega$ for $n = 0, 1, 2\ldots$

d. Show that the ground state energy is $E_0 = \frac{1}{2}\hbar\omega$ with normalized eigenstate

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{1}{2}\xi^2\right).$$

e. Show that the normalized eigenfunctions are

$$\psi_n = \frac{1}{\sqrt{n!(\hbar\omega)^n}}(a^\dagger)^n\psi_0.$$

3.

a. Show that the energy eigenstates for a 1-dimensional system with even potential function $V(x)$ can be chosen to be either even or odd functions in x.

b. Consider the energy eigenstates for a particle of mass m moving in the potential well

$$V(x) = \begin{cases} -\frac{\hbar^2}{m} < 0, |x| < \frac{\pi}{4} \\ 0, |x| > \frac{\pi}{4} \end{cases}$$

Show that this system has a ground state with energy $E = -\frac{\hbar^2}{2m}$ and find the normalized ground state.

c. Show that the ground state is the unique bound state.

4.

a. Prove the Schwarz inequality for states f, g in a Hilbert space

$$|\langle f, g\rangle| \leq \|f\| \cdot \|g\|,$$

where $\|f\| = \langle f, f\rangle$ etc.

b. Show for a given wavefunction ψ that $\sigma_x^2 = \|f\|^2$ for $f = (x - \langle x\rangle)\psi$ and $\sigma_p^2 = \|g\|^2$ for $g = (\hat{p} - \langle\hat{p}\rangle)\psi$

c. Hence prove the Heisenberg Uncertainty Principle for x and $\hat{p}$.

d. Show that the Gaussian wave packet

$$\psi(x) = A\exp(-a(x - \langle x\rangle)^2)\exp\left(\frac{ix\langle\hat{p}\rangle}{\hbar}\right).$$

is the unique solution with minimum value of $\sigma_x\sigma_p$ where A and $a > 0$ are constants. Express σ_x and σ_p in terms of a.

5. Consider a particle of mass m moving in an infinite spherical potential well with potential

$$V(r) = \begin{cases} 0, & 0 \leq r < a \\ \infty, & r > a \end{cases}$$

a. Show that the s-wave energy eigenfunctions are given by

$$\psi_n(r) = \frac{1}{\sqrt{2\pi a}}\frac{1}{r}\sin\left(\frac{n\pi r}{a}\right).$$

b. Find the energy eigenvalues.

c. Show that the probability of finding m in the region $r < a/2$ is one half.

You may assume that the Laplacian in spherical polar coordinates is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

6. Consider the angular momentum operators L_1, L_2, L_3 obeying the commutation relations

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k.$$

Define the ladder operators $L_\pm = L_1 \pm iL_2$.

a. Show that the following commutation relations hold

$$[L^2, L_3] = 0,$$
$$[L_3, L_\pm] = \pm\hbar L_\pm.$$

b. Show that $L^2 = L_\pm L_\mp + L_3^2 \mp \hbar L_3$.

c. Hence show that the L^2 and L_3 eigenstates $|lm\rangle$ obey

$$L^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle,$$
$$L_3|lm\rangle = \hbar m|lm\rangle,$$

for $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ with $m = -l, -l+1, \ldots, l-1, l$.

d. Show that orthonormalized eigenstates $|lm\rangle$ obey

$$L_\pm|lm\rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\,|lm \pm 1\rangle$$

7. A two particle system consists of spin $s_1 = \frac{3}{2}$ particle together with a spin $s_2 = \frac{1}{2}$ particle. Describe the total spin eigenstates $|sm\rangle$ of the system in terms of the basis $\{\ |s_1 m_1\rangle\,|s_2 m_2\rangle\}$ and hence find the Clebsch-Gordan coefficients $C^{s_1 s_2 s}_{m_1 m_2 m}$.

You may assume the relation $L_\pm|lm\rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\,|lm \pm 1\rangle$ for orthonormal simultaneous eigenstates of L^2 and L_3.

8. A Hamiltonian H^0 has orthonormal non-degenerate eigenstates ψ_n^0 for energy E_n^0. Let $H = H^0 + \lambda H'$ be a perturbed Hamiltonian for small λ.

a. Show to first order in λ that E_n^0 and ψ_n^0 are perturbed to

$$E_n = E_n^0 + \lambda\langle\psi_n^0, H'\psi_n^0\rangle,$$
$$\psi_n = \psi_n^0 + \lambda\sum_{m \neq n}\frac{\langle\psi_m^0, H'\psi_n^0\rangle}{E_n^0 - E_m^0}\psi_m^0$$

b. Verify the result for E_n for the Harmonic Oscillator Hamiltonian $H^0 = \frac{\hat{p}^2}{2m} + \frac{m\omega x^2}{2}$ with $H' = \lambda x^2$.

You may assume that $aa^\dagger = \hat{H} + \frac{\hbar\omega}{2}$ and $a^\dagger a = \hat{H} - \frac{\hbar\omega}{2}$ for $a = \frac{1}{\sqrt{2m}}(i\hat{p} + m\omega x)$.

9. Consider a system consisting of two spin $\frac{1}{2}$ particles of spin $\mathbf{S}^{(1)} = \frac{\hbar}{2}\sigma^{(1)}$ and $\mathbf{S}^{(2)} = \frac{\hbar}{2}\sigma^{(2)}$ and with zero total spin where $\sigma = \sigma_1\mathbf{i} + \sigma_2\mathbf{j} + \sigma_3\mathbf{j}$ for Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$.

a. Show for any two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ for $P(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \langle \hat{\mathbf{a}}.\sigma^{(1)}\hat{\mathbf{b}}.\sigma^{(2)} \rangle$ that

$$P(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = -\hat{\mathbf{a}}.\hat{\mathbf{b}}.$$

b. Suppose there is a hidden variable λ with density function $\rho(\lambda)$ such that the spin measurements are described by functions $A(\hat{\mathbf{a}}, \lambda)$ and $B(\hat{\mathbf{b}}, \lambda)$ respectively taking values ± 1. For $P(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \int \rho(\lambda) A(\hat{\mathbf{a}}, \lambda) B(\hat{\mathbf{b}}, \lambda) d\lambda$ prove Bell's inequality

$$\left| P(\hat{\mathbf{a}}, \hat{\mathbf{b}}) - P(\hat{\mathbf{a}}, \hat{\mathbf{c}}) \right| \leq 1 + P(\hat{\mathbf{b}}, \hat{\mathbf{c}}).$$

c. Demonstrate that Bell's inequality fails for the quantum mechanical prediction for $P(\hat{\mathbf{a}}, \hat{\mathbf{b}})$. Comment on the significance of this result.

Useful formulae

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}, \text{ 1-d Momentum Operator}$$

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(x), \text{ 1-d Hamiltonian Operator}$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t), \text{ 1-d Schrödinger Equation}$$