Ollscoil na hÉireann, Galway GX 2178
National University of Ireland, Galway

Semester II Examinations, 2003/2004

Exam Code(s)	3CS1,3CS2,1EM1,3BS3,3BS9,3EL1,3EL2
Exam(s)	Third Science
Module Code(s)	MP364
Module(s)	Methods of Mathematical Physics II
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Micheál Ó Confhaola
	Dr. M.G. Meere

Instructions: Attempt **THREE** questions

Duration	**TWO HOURS**
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	2 (excluding this page)
Department(s)	Mathematical Physics

1. Consider the following initial boundary value problem for a function $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0,$$
$$\frac{\partial u}{\partial x} = 0 \text{ on } x = 0, t > 0,$$
$$u = 0 \text{ on } x = 1, t > 0,$$
$$u = f(x) \text{ at } t = 0, 0 < x < 1,$$

where $f(x)$ is a given smooth function and α^2 is a positive constant. Solve this problem using the method of separation of variables. What is the solution if $f(x) = x$?

2. Consider the following two point boundary value problem for a function $y(x)$:

$$L(y) = \lambda r(x)y,\ 0 < x < 1,$$
$$a_1 y(0) + a_2 y'(0) = 0,\ b_1 y(1) + b_2 y'(1) = 0$$

where

$$L(y) = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y$$

and where $p(x), q(x), r(x), a_1, a_2, b_1, b_2$ satisfy all of the usual conditions for the above problem to be regular.

   a. If $u(x), v(x)$ are any pair of twice continuously differentiable real valued functions, then show that

$$(L(u), v) - (u, L(v)) = -p(x)(u'(x)v(x) - u(x)v'(x))\Big|_0^1,$$

   where $(u,v) = \int_0^1 u(x)v(x)dx$ is the usual inner product. If, in addition, $u(x), v(x)$ satisfy the boundary conditions on $x = 0, 1$ given above, then show that (you may assume that $a_2 b_2 \neq 0$)

$$(L(u), v) = (u, L(v)).$$

   b. Determine the normalized eigenfunctions of

$$y'' + \lambda y = 0,\ y(0) = y(1) = 0.$$

   Find the expansion of $f(x) = x^2$ in terms of these eigenfunctions.

3. Consider the following boundary value problem for Laplace's equation in a cylindrical region:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0 \text{ in } 0 \le \rho < a,\ 0 < z < b,$$
$$u = 0 \text{ on } \rho = a,\ 0 < z < b,$$
$$u = 0 \text{ on } z = 0,\ 0 \le \rho < a,$$
$$u = f(\rho) \text{ on } z = b,\ 0 \le \rho < a,$$

where $u = u(\rho, z)$, a, b are positive constants and $f(\rho)$ is a given smooth function.

   a. Letting $u(\rho, z) = R(\rho)Z(z)$, show that

$$\rho^2 R'' + \rho R' + \lambda \rho^2 R = 0 \text{ and } Z'' - \lambda Z = 0$$

   where λ is the separation constant.

   b. Construct the eigenfunctions for the boundary value problem, and hence obtain the solution for u.

   Note. The equation $\rho^2 R'' + \rho R' + \lambda \rho^2 R = 0$ has the general solution $R(\rho) = AJ_0(\sqrt{\lambda}\rho) + BY_0(\sqrt{\lambda}\rho)$ where A, B are arbitrary constants and J_0, Y_0 are Bessel functions of order zero of the first and second kind, respectively.

4. Consider the following initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0,$$

$$u = f(x) \text{ at } t = 0, -\infty < x < \infty$$

where $u = u(x,t)$, K is a positive constant and $f(x)$ is a given smooth function. The intervals $-\infty < x < \infty$ and $t > 0$ are split into the uniform grids

$$\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots \text{ and } 0 = t_0 < t_1 < t_2 < \ldots$$

where $x_j = j(\Delta x), j = \ldots,-2,-1,0,1,2,\ldots$ and $t_k = k(\Delta t), k = 0,1,2,\ldots$ with a view to solving the problem numerically using finite difference techniques. The numerical approximation to u at gridpoint (x_j, t_k) is denoted by U_j^k.

a. Using standard finite difference approximations for the derivatives, derive the following implicit time-stepping scheme for solving the above problem:

$$-KhU_{j-1}^k + (1 + 2Kh)U_j^k - KhU_{j+1}^k = U_j^{k-1},$$

where $h = \Delta t/(\Delta x)^2$.

b. Denote by $\bar{e}_j^k$ the error in the implementation of the above difference scheme; these errors also satisfy the scheme. By considering a representative Fourier term $\bar{e}_j^k = \exp(i(\theta j + \lambda k))$ ($i^2 = -1$) where θ and λ are constants, show that

$$\exp(-i\lambda) = 1 + 4Kh\sin^2(\theta/2),$$

and hence show that the implicit scheme is von Neumann stable for all $h > 0$.

c. Derive an explicit scheme for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$

where $u = u(x,t)$ and c is constant, using standard central difference approximations for the second derivatives. Derive a stability criterion for the scheme.

5. Consider the following initial boundary value problem for the wave equation on an infinite domain:

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty,\ t > 0,$$

$$u = 4\exp(-5|x|) \text{ at } t = 0,\ -\infty < x < \infty,$$

$$\frac{\partial u}{\partial t} = 0 \text{ at } t = 0,\ -\infty < x < \infty,$$

where $u = u(x,t)$. You may assume that u and its derivatives tend to zero as $x \to \infty$. Solve this problem by taking Fourier transforms in the x variable.

Notes: (i) The Fourier transform and Fourier inverse transform of a function $f(x)$, $-\infty < x < \infty$, are

$$\overline{f(c)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{icx}f(x)dx \text{ and } f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-icx}\overline{f(c)}dc.$$

(ii) You may assume that the inverse transform of

$$\sqrt{\frac{2}{\pi}}\,\frac{20}{25 + c^2}\cos(3ct)$$

is given by

$$2\exp(-5|x + 3t|) + 2\exp(-5|x - 3t|).$$