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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER II EXAMINATIONS 2004/2005

MA378 – NUMERICAL ANALYSIS II

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Time allowed: **Two** hours.
Attempt **THREE** questions.

Q1. Suppose that $f(x)$ is a real-valued, continuous function on $[a,b]$. Let $p_n(x)$ be a polynomial of degree n that interpolates the function $f(x)$ at the distinct points $a = x_0, x_1, \ldots, x_n = b$.

(a) Such that a polynomial of degree at most n with $n+1$ zeros is in fact zero everywhere. Hence show that the interpolating polynomial $p_n(x)$ is unique.

(b) Show that $p_n(x)$ exists.

(c) Write down the Lagrange form of $p_2(x)$, the polynomial of degree 2 that interpolates $f(x) = \ln(x^2)$ at $x_0 = 1, x_1 = 2$ and $x_2 = 3$.

(d) Use the fact that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x), \qquad \text{for some } \tau = \tau(x) \in (a,b), \tag{1}$$

where $\pi_{n+1}(x)$ is the nodal polynomial, to find an upper bound for the interpolation error in (c) at $x = 3/2$.

Q2. Let $l(x)$ be the *piecewise linear* interpolant to $f(x)$ on $a = x_0 < x_1 < \cdots < x_n = b$.

(a) Define $l(x)$, and give a formula for it.

(b) Let $h = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n$. Derive an error estimate for $\|f(x) - l(x)\|_{\infty,[a,b]}$. (You may assume the formula (1) above.)

(c) Suppose a linear spline is used to interpolate $f = 1/(1+x^2)$ on $n+1$ equally spaced points on the interval $[1,5]$. Find the smallest value of n you would have to take to ensure that $\|f(x) - l(x)\|_{\infty,[1,5]} \leq 10^{-2}$.

Q3. (a) Define the Newton-Cotes Quadrature Rules for estimating definite integrals. What is meant by the *precision* of a Quadrature rule?

(b) Derive the three-point Newton-Cotes formula (*Simpson's Rule*).

(c) If one uses (1) above to derive an error estimate for Simpson's Rule it would suggest that the rule has precision 2. Show that it has in fact precision 3.

(d) Derive the *Composite Simpson's Rule*. Use it to approximate

$$\int_0^2 e^{-x^2}\,dx,$$

on 5 equally spaced points.

Q4. (a) Find A_0, A_1, x_0 and x_1 such that the two-point rule

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx G_2(f) := A_0 f(x_0) + A_1 f(x_1)$$

computes the integral exactly for any polynomial of degree 3 or less.

(b) Let $\{\hat{p}_n(x)\}_{n=0}^{\infty}$ be the sequence of orthogonal monic polynomials with respect to the inner product

$$(f, g) := \int_a^b f(x)g(x)\mathrm{d}x.$$

Show that

(i) $\hat{p}_n(x)$ is orthogonal to *all* polynomials of degree less than n.

(ii) the zeros of the $\hat{p}_n(x)$ are simple (distinct).

(iii) the zeros of the $\hat{p}_n(x)$ are all in the interval $[a, b]$.

Q5. Consider the Boundary Value Problem: *find* $u(x) \in C^2(0,1)$ *such that*

$$-u''(x) + a(x)u(x) = f(x) \quad \text{on } (0,1), \qquad u(0) = u(1) = 0.$$

(a) Write down the variational formulation of this problem, and give a short description of the Galerkin Finite Element Method with piecewise linear basis functions.

(b) If the FEM is formulated as *find* $u^h \in S_0^h$ such that

$$\mathcal{A}(u^h, v^h) = (f, v^h) \qquad \text{for all} \qquad v^h \in S_0^h,$$

prove *Cea's Lemma*: that

$$\mathcal{A}(u - u^h, v^h) = 0 \qquad \text{for all} \qquad v^h \in S_0^h,$$

and

$$\mathcal{A}(u - u^h, u - u^h) = \min_{v^h \in S_0^h} \mathcal{A}(u - v^h, u - v^h).$$