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OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2005

B.A. and B.Sc. Degree Examination

MA391 - Probability

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Time allowed: *two* hours.
Answer *three* questions.

1. (a) The number X of phone calls received by a typical garda station in a given hour has $f_X(x) = \lambda^x e^{-\lambda}/x!$ $(x = 0, 1, 2, \ldots)$. Let Y be the time a garda has to wait between arriving on duty and receiving the first call. Show that Y has density function $f_Y(t) = \lambda e^{-\lambda t}$ $(t \geq 0)$.

   (b) A garda station receives an average of 3 phone calls per hour. What is the probability that a garda receives his first phone call during the second hour of his duty shift?

   (c) A point is chosen uniformly from one side of an equilateral triangle with sides of length 2. Let X be the distance from the point to the opposite vertex. Find the distribution function of X. Also, express the expected value of $1/X$ as a definite integral.

2. (a) Suppose that X, Y are random variables with joint density function $f_{X,Y}$ given by the following table.

X \ Y	-1	0	2	6
-2	1/9	1/27	1/27	1/27
1	2/9	0	4/27	?
3	0	0	1/9	4/27

PTO

i. Find the marginal distribution function $F_X(x)$.

ii. Are X and Y independent? (Justify your answer.)

iii. Find the probability that XY is odd.

iv. Find the conditional density function $f_{X|Y}(x|2)$ for X given that $Y = 2$.

v. Find the expected value of $X^2 + Y^2$.

vi. Find the expected value of $X^2 + Y^2$ given that $Y = 2$.

(b) The times (in minutes) it takes Peter and Paul to solve a problem are independent and exponentially distributed with parameter λ. Find the probability that Peter will take at least twice as long as Paul to solve the problem. Also, express the expected value of the sum of their times as a double integral.

3. (a) State the Central Limit Theorem.

(b) An opinion poll is to be carried out to determine the number of people who would currently vote 'yes' in the forthcoming referendum on the EU constitution. We would like to be 95% confident that the poll's estimate is within 3% of the actual number. How many people would you poll? Justify you answer and state any assumptions you make.

(c) Prove the Central Limit Theorem. You may assume the Continuity Theorem and that

$$\lim_{n\to\infty} n(\log \phi_{X_1}(t/\sigma\sqrt{n}) - \mathrm{i}\mu(t/\sigma\sqrt{n})) = -t^2/2$$

for any random variable X_1 with mean μ and non-zero variance σ^2.

4. Attempt only **three** of the following.

(a) Suppose that the weight X of a random person is normally distributed, that 50% of people weigh less than 80 kilos, and 25% weigh less than 70 kilos. What percentage weigh over 90 kilos?

(b) Let $f(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$ and $g(x) = \lambda^x e^{-\lambda}/x!$. Prove that, for a positive integer x and for a fixed $\lambda > 0$ with $\theta = \lambda/n$, $\lim_{n\to\infty} f(x) = g(x)$.

(c) Define a *probability density function* and prove that $f(t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(t-\mu)^2/2\sigma^2}$ is such a function for any constants $\sigma > 0$ and μ.

(d) Let X be a random variable with moment generating function $M_X(t) = e^{\mu t}e^{\sigma^2 t^2/2}$. Calculate the mean and variance of X. Also, for $\mu = 0$, calculate the moments $E(X^n)$ and the characteristic function $\phi_X(t)$.

(e) Let X, Y be independent random numbers uniformly distributed on $[0, 2]$. Determine the expected distance between X and Y.