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OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2005 - HONOURS

B.A. and B.Sc. EXAMINATIONS
HIGHER DIPLOMA IN MATHEMATICS

MATHEMATICS

MA416 [RING THEORY] and MA491 [FIELD THEORY]

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Time allowed: **Three** hours.

Those seeking credit for one semester should answer *three* questions.
Those seeking credit for both semesters should answer *five* questions.
Please use separate answer books for each section.

SECTION A — RING THEORY

1. Let R be a commutative ring with an identity 1.

(a) Give the definition of an ideal I of R.

(b) When I is an ideal of R, describe the factor ring R/I. Give the zero and identity of the factor ring.

(c) When $R = Z_2[x]$ and $I = Ra$, where $a = x^3 + x + 1$. Show that R/I has eight elements, and that each non-zero element has a multiplicative inverse. Show also that the multiplicative group is cyclic.

p.t.o.

2. (a) Give the definition of a ring homomorphism α from a ring R into a ring S.

(b) Show that $Im(\alpha) = \{\alpha(r) : r \in R\}$ is a subring of S.

(c) When α is a ring homomorphism from $Z \times Z$ into $Z \times Z$, find the possible images for $(1,0)$ and $(0,1)$. List all the ring homomorphisms from $Z \times Z$ into $Z \times Z$.

3. Let R be a principal ideal domain.

(a) Give the definition of an irreducible element p in R.

(b) Let $0 \neq M = Rp$ be a maximal ideal in R. Prove that p is irreducible.

(c) Assuming that every ideal Ra of R is contained in some maximal ideal, when $0 \neq a$ and a is not a unit, prove that every such element a is divisible by an ireducible element.

(d) What are the irreducible elements in the ring of integers?

4. (a) State and prove Eisenstein's Criterion for the irreducibility of a polynomial in $Z[x]$.

(b) Show that the polynomial $2x^5 - 6x^3 + 9x + 15$ has no rational roots.

Section B – Field Theory (MA 491)

B1. (i) Explain how a field $\mathbf{K}$ may be viewed as a vector space over a sub–field $\mathbf{F}$ and hence define the degree $[\mathbf{K} : \mathbf{F}]$.
(ii) Define the term **splitting field** of a polynomial. State the roots of $x^4 + 1$ and calculate the degree of its splitting field over $\mathbf{Q}$.
(iii) Find the minimum polynomial of $i + \sqrt{2}$ over $\mathbf{Q}$.
(iv) Show that $x^4 + 1$ is *reducible* over $\mathbf{Q}(\sqrt{2})$, but does not split over $\mathbf{Q}(\sqrt{2})$. Deduce that $K = \mathbf{Q}(i + \sqrt{2})$ is the splitting field of $x^4 + 1$ over $\mathbf{Q}$.

B2. (i) What is meant by a "straight–edge and compass construction"?
(ii) State Gauss' Theorem concerning the values for which the regular n–gon can be constructed by straight edge and compass.
(iii) Using the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, or otherwise, **prove** that the angle $10°$ is not constructible.
(iv) Using part (ii) show that $18°$ is constructible. Deduce that the angle $n°$ is constructible $\Leftrightarrow 3|n$.
(v) Show that $\cos^{-1}\left(\frac{23}{27}\right)$ can be trisected using straight–edge and compass.

B3. (i) Let p be a prime. Show that $x^p - 2$ is irreducible over $\mathbf{Q}$.
Prove that the splitting field of $x^p - 2$ over $\mathbf{Q}$ has degree $p(p-1)$.
(ii) Determine the Galois group G of $x^3 - 2$ over $\mathbf{Q}$ and establish that G is non–abelian of order 6.
(iii) Under the Galois correspondence find the (fixed) subfield corresponding to the subgroup of G of order 3.

B4. (i) Let $\mathbf{GF}(q)$ be a finite field of order q $(= p^n$, p a prime $n \geq 1)$. State the main properties of $\mathbf{GF}(q)$.
(ii) Prove that $\mathbf{GF}(q)$ is the splitting field of $x^{p^n} - x$ over $\mathbf{Z}_p$.
(iii) Let $f(x)$ be a monic irreducible polynomial of degree m over $\mathbf{Z}_p$.

Prove that $f(x)$ divides $x^{p^n} - x \iff m|n$. Hence or otherwise deduce that

$$p^n = \sum_{d|n} dN_p(d)$$

where $N_p(d)$ is the number of monic irreducible polynomials of degree d over $\mathbf{Z}_p$.

(iv) Calculate $N_2(1)$, $N_2(2)$, $N_2(4)$ and hence, or otherwise, factorise $x^{16} - x$ into irreducible factors over $\mathbf{Z}_2$.