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OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2005

B.A. and B.Sc. Degree Examination

MATHEMATICS [MA418]

MA418 — *DIFFERENTIAL EQUATIONS WITH FINANCIAL DERIVATIVES*

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Time allowed: *Two* hours.
Answer <u>three</u> questions.

Q1. (a) Prove that *Standard Brownian motion* B_t is a Gaussian process, and hence prove that $(T^{1/2}B_{t_1}, \ldots, T^{1/2}B_{t_n}) = (B_{Tt_1}, \ldots, B_{Tt_n})$ for every $T > 0$, $t_i > 0$, $i = 1, \ldots, n$, and $n > 0$.

(b) Prove that B_t is a.s. non-differentiable at $t = 0$.

(c) Prove that B_t is a.s. non differentiable at every t, $t \geq 0$.

Q2. (a) Prove that $e^{B_t}e^{-t/2}$ is an $\mathcal{F}_t$ martingale, where $\mathcal{F}_t$ is the natural filtration for Brownian motion.

(b) Let $\Omega = [0, 1]$ with the σ- field of Borel sets, and let P be Legesgue measure on $[0, 1]$. Consider the random variables $X(\omega) = 2\omega^2$, $Y(\omega) = 1 - |2\omega - 1|$, $\omega \in [0, 1]$. Describe $\sigma(Y)$, and calculate $E(X|\sigma(Y))$.

p.t.o.

Q3. (a) In the Vasicek model the *instantaneous interest rate* r_t satisfies;

$$dr_t = c[\mu - r_t]dt + \sigma dB_t, \ \ t \in [0, T],$$

where c, μ and σ are positive constants. Use the Ito Lemma to solve this stochastic differential equation and hence prove that;

$$E[r_t] = r_0 e^{-ct} + \mu(1 - e^{-ct}), \quad r_0 \in \mathbb{R}.$$

(b) Consider a self-financing strategy (a_t, b_t), and associated value process $V_t = a_t X_t + b_t \beta_t = u(T - t, X_t)$, $t \in [0, T]$. Assuming the stock X_t and bond β_t satisfy $dX_t = cX_t dt + \sigma X_t dB_t, \ c, \sigma > 0$, and $d\beta_t = r\beta_t dt, \ r > 0$, respectively, derive the Black-Scholes partial differential equation;

$$u_1(t, x) = \frac{1}{2}\sigma^2 x^2 u_{22}(t, x) + rxu_2(t, x) - ru(t, x)$$

$x > 0, t \in [0, T]$.

Q4. (a) State Girsanov's (change of measure) theorem. Assume that in the Black-Scholes model there exists a self-financing strategy (a_t, b_t) such that the value V_t of the portfolio at time t is $V_t = a_t X_t + b_t \beta_t, \ \ t \in [0, T]$, and V_T is equal to the contingent claim $h(X_T)$. Prove that

$$V_t = E_Q[e^{-r(T-t)} h(X_T) | \mathcal{F}_t]$$

where $\mathcal{F}_t$ is the natural filtration for Brownian motion, and Q is the equivalent martingale measure in Girsanov's theorem which converts $B_t + \frac{(c-r)}{\sigma} t$ into standard Brownian motion.

(b) Prove that the European put price P_T and the European call price C_T (with strike price K) are related by the formula $P_T = C_T - X_0 + Ke^{-rT}$ where the stock price satisfies $dX_t = cX_t dt + \sigma X_t dB_t, \ c, \sigma > 0$.