978

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATION, 2005

FOURTH COMPUTER SCIENCE AND FOURTH SCIENCE EXAMINATION

FOURTH YEAR MATHEMATICS OPTION

[MA426]

Dr. Dave Johnson
Prof. T. Hurley
Dr. J. McDermott

Time allowed: *Two* hours
Full marks for *three* questions

1. (a) The *mixed strategies* $\mathbf{r}$ and $\mathbf{s}$ are *optimal* for the players R and C in the *zero-sum* game defined by the matrix A, and $v_R \leq E(\mathbf{r},\mathbf{s}) \leq v_C$. Explain the terms and symbols here, justify the inequalities, and state von Neumann's theorem. Show that if $\mathbf{p}$ and $\mathbf{q}$ are strategies such that $\min \mathbf{p}A = \max A\mathbf{q}' = k$ then $\mathbf{p}$ and $\mathbf{q}$ are optimal strategies (for R and C) and $v = k$ is the value of the game.

(b) Consider the zero-sum game defined by the matrix

$$A = \begin{pmatrix} 4 & -3 & -2 \\ 0 & -4 & 1 \\ -1 & 3 & 2 \end{pmatrix}.$$

Show that R_2 is dominated by a suitable combination of R_1 and R_3. Solve the resulting 2×3 game in detail, and hence solve A. Check your solutions using the criterion at the end of (a) above.

2. R and C are playing the zero-sum game defined by a matrix A^+ in which $a_{ij} > 0$ for all i and j. Explain how R's aim leads to a linear programming problem (LPP), and C's aim leads to the dual LPP.
Use the tableau (simplex) method to solve the pair of dual LPPs that correspond to the matrix A^+ below left, and hence solve the game defined by A below right.

$$A^+ = \begin{pmatrix} 5 & 2 & 6 \\ 1 & 4 & 3 \\ 5 & 3 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -2 & 2 \\ -3 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

3. (a) Prove the 2×2 case of Nash's Theorem: every non-zero-sum (NZS) game has a Nash equilibrium (NE).
(b) Find all the NEs in the game defined by
$$A = \begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$
and decide which (if any) is Pareto optimal. Is this game strictly soluble? Why (not)?
(c) Consider the generic 2×2 NZS games in which each of A and B contains the numbers $4,3,2,1$ in some order. Determine, with explanation, the number of essentially distinct such games.

4. (a) A sequence of rounds of the game below is played, the n^{th} round occurring with probability p^{n-1} for some fixed p in $(0,1)$. R is considering two strategies: I R_1 always; II R_1 twice, then R_2 always - and she knows that C always uses tit-for-tat (C_1 first, then C_j if R played R_j in the previous round). For which values of p would I be better than II for R? What would C like?
$$A = \begin{pmatrix} 3 & 1 \\ 7 & -2 \end{pmatrix}, \qquad B = A' = \begin{pmatrix} 3 & 7 \\ 1 & -2 \end{pmatrix}.$$

(b) Give the definition of an evolutionarily stable strategy (ESS). Describe briefly the ideas that motivate the definition, and determine the unique ESS in the above game.