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OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2005

B.Sc. (Part II) EXAMINATION
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS — [MA490]

MEASURE THEORY

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Time allowed: **Two** hours.
Answer *three* questions.

1. Answer *four* of the following:

   (a) In Bernoulli space, let E be the event that a sequence of coin tosses eventually produces HTHT. Express E in terms of the simple events E_n (E_n is the event that the nth toss is H.)

   (b) Give the construction of the Cantor set C and show that $m^*(C) = 0$.

   (c) If f is a measurable function on a measure space, show that the function $|f|$ is also measurable. Is the converse true? Explain your answer.

   (d) The σ-algebra $\mathcal{A}$ of subsets of $\mathbb{R}$ consists of the following sets: $\varnothing$, $\mathbb{R}$, $(-\infty, 1)$, $[1, \infty)$. Describe all the functions that are measurable with respect to $\mathcal{A}$.

   (e) Evaluate the following expression by taking the summation inside the integral. Justify your calculation by appealing to a suitable convergence theorem.

   $$\sum_{n=0}^{\infty} \int_{\pi/4}^{\pi/2} (1 - \sin^3 x)^n \cos x \, dx \, .$$

   (f) Describe briefly the relationship between the Riemann and Lebesgue integrals for bounded functions on an interval $[a, b]$. Give an example of a function that is Lebesgue integrable but not Riemann integrable.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) If (E_n) is a decreasing sequence of sets in $\mathcal{A}$ with intersection E and if $\mu(E_1) < \infty$, show that
$$\mu(E) = \lim_{n\to\infty} \mu(E_n)\,.$$
Show by an example that this result may fail if the condition $\mu(E_1) < \infty$ is omitted.

(b) Let (E_n) be a sequence of measurable subsets of X. Give the definition of the set $\limsup E_n$ and show that
$$\limsup \mu(E_n) \le \mu\big(\limsup E_n\big)\,.$$

(c) Suppose that (E_n) is a sequence of measurable events in a probability space with the property that $P(E_n) \to 1$ as $n \to \infty$. What can you say about the probability of the event that E_n occurs infinitely often?

3. (a) Give the definition of a *measurable function*. Show that if f and g are measurable then so is the function $f + g$.

(b) Let (f_n) be a sequence of measurable functions that converges pointwise to a function f. Show that f is measurable.

(c) Let f and g be functions on a complete measure space. If f is measurable and $f = g$ almost everywhere, show that g is measurable.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give an account (without proofs) of the definition of the integral $\int_X f \, d\mu$, starting with the integral of a simple measurable function.

(b) Let f be a nonnegative measurable function with the property that

$$\int_E f \, d\mu = 0 \quad \text{for every measurable set } E\,.$$

Show that $f = 0$ almost everywhere.

(c) State and prove **either** the Lebesgue Monotone Convergence Theorem **or** the Lebesgue Dominated Convergence Theorem.

5. (a) Give the definition of the *Lebesgue Outer Measure* $m^*(E)$ of a subset E of $\mathbb{R}$. What is meant by saying that E is *Lebesgue measurable*? Show that, if (E_n) is a sequence of Lebesgue measurable sets, then

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^*(E_n)\,.$$

(b) Describe the construction, with proof, of a subset E of $\mathbb{R}$ that is not Lebesgue measurable.