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OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2005

B.Sc. (Part II) EXAMINATION
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS — [MA490/482]

MEASURE THEORY AND FUNCTIONAL ANALYSIS

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Time allowed: **Three** hours.
Full marks for five questions.
Use separate answer books for each section.

SECTION A — *MEASURE THEORY*

A1. Answer *four* of the following:

(a) In Bernoulli space, let E be the event that a sequence of coin tosses eventually produces HTHT. Express E in terms of the simple events E_n (E_n is the event that the nth toss is H.)

(b) Give the construction of the Cantor set C and show that $m^*(C) = 0$.

(c) If f is a measurable function on a measure space, show that the function $|f|$ is also measurable. Is the converse true? Explain your answer.

(d) The σ-algebra $\mathcal{A}$ of subsets of $\mathbb{R}$ consists of the following sets: $\varnothing$, $\mathbb{R}$, $(-\infty, 1)$, $[1, \infty)$. Describe all the functions that are measurable with respect to $\mathcal{A}$.

(e) Evaluate the following expression by taking the summation inside the integral. Justify your calculation by appealing to a suitable convergence theorem.

$$\sum_{n=0}^{\infty} \int_{\pi/4}^{\pi/2} (1 - \sin^3 x)^n \cos x \, dx \,.$$

(f) Describe briefly the relationship between the Riemann and Lebesgue integrals for bounded functions on an interval $[a, b]$. Give an example of a function that is Lebesgue integrable but not Riemann integrable.

A2. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) If (E_n) is a decreasing sequence of sets in $\mathcal{A}$ with intersection E and if $\mu(E_1) < \infty$, show that

$$\mu(E) = \lim_{n\to\infty} \mu(E_n)\,.$$

Show by an example that this result may fail if the condition $\mu(E_1) < \infty$ is omitted.

(b) Let (E_n) be a sequence of measurable subsets of X. Give the definition of the set $\limsup E_n$ and show that

$$\limsup \mu(E_n) \leq \mu(\limsup E_n)\,.$$

(c) Suppose that (E_n) is a sequence of measurable events in a probability space with the property that $P(E_n) \to 1$ as $n \to \infty$. What can you say about the probability of the event that E_n occurs infinitely often?

A3. (a) Give the definition of a *measurable function*. Show that if f and g are measurable then so is the function $f + g$.

(b) Give an account (without proofs) of the definition of the integral $\displaystyle\int_X f\, d\mu$, starting with the integral of a simple measurable function. Give an example of a measurable function that is not integrable

(c) State and prove **either** the Lebesgue Monotone Convergence Theorem **or** the Lebesgue Dominated Convergence Theorem.

A4. (a) Give the definition of the *Lebesgue Outer Measure* $m^*(E)$ of a subset E of $\mathbb{R}$. What is meant by saying that E is *Lebesgue measurable*? Show that, if (E_n) is a sequence of Lebesgue measurable sets, then

$$m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} m^*(E_n)\,.$$

(b) Describe the construction, with proof, of a subset E of $\mathbb{R}$ that is not Lebesgue measurable.

SECTION B — *FUNCTIONAL ANALYSIS*

B1. (a) Show $\|x\|_{l^8} \leqq \|x\|_{l^7}$ for all $x \in l^7$.

(b) Let X and Y be normed spaces with $T : X \to Y$ a linear mapping. If T is continuous at 0 show there exists $M > 0$ with $\|Tx\| \leqq M\|X\|$ for all $x \in X$.

(c) Let X be a normed space and Y a Banach space. Show $B(X,Y)$ is a Banach space.

B2. (a) Show that the dual space $(c_0)'$ is isometrically isomorphic to l^1.

(b) A linear functional is defined on the Banach space l^8 by $\phi(x) = 2x_1 - 3x_2 + 7x_4 - 5x_6$ where $x = (x_1, x_2, ...) \in l^8$. Find the norm of ϕ.

B3. (a) Let X be a normed space and $x_0 \in X$ with $x_0 \neq 0$. Use the Hahn-Banach theorem to show that there exists a bounded linear functional ϕ on X with $\|\phi\| = \frac{1}{\|x_0\|}$ and $\phi(x_0) = 1$.

(b) Show c_0 (with the usual sup norm) is not a Hilbert Space.

(c) Let K be a nonempty closed convex subset of a Hilbert space H. Show for every $x \in H$ there is a unique $y \in K$ with

$$\|x - y\| = \inf_{z \in K} \|x - z\|.$$

B4. (a) Suppose H_1 and H_2 are Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. Define the Hilbert adjoint T^* of T. Show T^* exists, is unique, and is a bounded linear operator with norm $\|T^*\| = \|T\|$.

(b) Let $S : l^2 \to l^2$ be given by

$$S(x) = (0,\, x_1,\, x_2,\, ...) \text{ when } x = (x_1, x_2, ...)$$

For $n \in \{1, 2, ...\}$ show $S^*(e_{n+1}) = e_n$.