9§2

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2004/05

FOURTH UNIVERISTY EXAMINATION

STOCHASTIC PROCESSES - MA494

Dr. Dave Johnson
Professor T. Hurley
Dr. M. Hayes

Time allowed: *Two* hours.

In addition to this paper you should have available an electronic calculator not capable of storing text and logarithmic tables.

ATTEMPT THREE QUESTIONS.
ALL QUESTIONS HAVE EQUAL MARKS.

pto

1. (a) (i) Define the Markov property mathematically when applied to a discrete parameter stochastic process.

(ii) Explain the concept of Markov chains having stationary transition probabilities.

(b) If X_n, $n \geq 0$ is a two state Markov chain with

$$P(X_{n+1} = 1 | X_n = 0) = p,$$

$$P(X_{n+1} = 0 | X_n = 1) = q,$$

find:

(i)

$$P(X_1 = 0 | X_0 = 0, X_2 = 0),$$

(ii)

$$P(X_1 \neq X_2)$$

if $P(X_0 = x_0) = \pi_0(0)$,

(iii)

$$P_0(T_1 = n),$$

(iv)

$$P_1(T_0 = n).$$

Show all your work.

pto

2. (a) Define:

(i) recurrent state,

(ii) transient state.

(b) For arbitary states x and y prove that

$$P_x(N(y) = m) = \rho_{xy}\rho_{yy}^{m-1}(1 - \rho_{yy}),$$

where $N(y)$ is the number of times that the chain is in state y.

(c) Consider a Markov chain on { 1, 2, 3, 4, 5, 6} having transition matrix:

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ 2 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 4 & \frac{1}{10} & \frac{1}{10} & 0 & 0 & \frac{2}{5} & \frac{2}{5} \\ 5 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 6 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \end{array}$$

(i) Determine which states are transient and which states are recurrent. Explain your answer fully.

(ii) Find $\rho_{\{1,2\}}(x)$ with $x = 2, 4, 6$.

pto

3. (a) Let X_n, $n \geq 0$, be a Markov chain whose state space H is a subset of $\{0, 1, 2, ...\}$ and whose transition function P is such that

$$\sum_y yp(x,y) = Ax + B \ , \ x \epsilon H \tag{1}$$

for some constants A and B.

(i) Show that $E[X_{n+1}] = AE[X_n] + B$.

(ii) Show that if $A \neq 1$, then

$$E[X_n] = \frac{B}{1-A} + A^n \left(E[X_0] - \frac{B}{1-A} \right).$$

(b) Let X_n, $n \geq 0$, be the Ehrenfest chain on $\{0, 1, 2, ...d\}$. Show that equation (1) in part (a) holds and use it to compute $E_x[X_n]$.

4. (a) In the context of Markov pure jump processes, define $F_x(t)$ and the transition probability Q_{xy}.

(b) The forward equation relating the probability that a Markov pure jump process starting at x will be at state y at time t to the infinitesimal parameters is

$$p'_{xy}(t) = \sum_z p_{xz}(t) q_{zy}.$$

The infinitesimal parameters q_{xy} are

$$q_{xy} = \begin{cases} -q_x & y = x \\ q_x Q_{xy} & y \neq x. \end{cases}$$

Find the transition function of the two-state birth and death process by solving the forward equation.