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OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2005

GROUPS II (MA533)

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Time allowed: *two* hours.
Answer *three* questions.

1. (a) Let A be a finite alphabet. Define the *free monoid* A^*, and prove that it is a monoid.

(b) What does it mean for a monoid M to *act* on a set X? Show that M acts on X if and only if there is a monoid homomorphism $f\colon M \to \operatorname{Fun}(X)$, where $\operatorname{Fun}(X)$ as usual denotes the full transformation monoid of all functions $X \to X$.

(c) Let $A = \{a, b\}$ and let A^* act on $X = \{0, 1, 2, 3\}$ according to

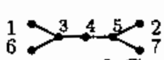
$$0a = 1b = 1, \quad 1a = 2b = 2, \quad 2a = 3a = 0b = 3b = 3.$$

(i) Draw a diagram of this action.
(ii) Let M be the submonoid of $\operatorname{Fun}(X)$ defined by $M = \{\tilde{w} \mid w \in A^*\}$, where $x\tilde{w} = xw$ for all $x \in X$. Determine all elements of M as words in the generators $\tilde{a}$, $\tilde{b}$. (M has precisely 6 elements.)
(iii) Describe the regular languages $L(3, 0) = \{w \in A^* \mid 3w = 0\}$ and $L(1, 3) = \{w \in A^* \mid 1w = 3\}$.

2. Let G be a group acting on a finite set X.

(a) Define the orbit xG of $x \in X$ and the stabilizer G_x. Prove that G_x is a subgroup of G.

(b) (i) Define a bijection between the sets xG and G/G_x. Deduce the Orbit-Stabilizer Formula: $|G| = |G_x| \cdot |xG|$, for all $x \in X$.
(ii) Use the bijection defined in (i) to prove that the action of G on xG is similar to an action of G on the coset space G/G_x.
(iii) What are the kernels of the actions in (ii)? i.e. which elements of G fix every element of xG, and which elements of G fix every element of G/G_x?

(c) Now suppose $G = \operatorname{Aut}\mathcal{G}$, where $\mathcal{G}$ is the graph 

, and X is the vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ of $\mathcal{G}$. Determine all elements of G and all orbits in X under the action of G.

p.t.o.

3. Let G be a finite group.

(a) Show that G acts on itself by right multiplication. Deduce Cayley's Theorem: G is isomorphic to a subgroup of $\mathrm{Sym}(n)$, where $n = |G|$. Construct an explicit isomorphism of $\langle a, b \mid a^2 = b^2 = (ab)^2 = 1\rangle$ into $\mathrm{Sym}(4)$.

(b) Suppose G acts on the finite set X. Let t be the number of orbits in X under the action of G. For each $a \in G$ define $\mathrm{Fix}\, a = \{x \in X \mid xa = x\}$. Prove that

$$t = \frac{1}{|G|} \sum_{a \in G} |\mathrm{Fix}\, a|.$$

(c) Let $\mathcal{G}$ be a graph and G be its automorphism group. Define an action of G on the set of colorings of the vertices of $\mathcal{G}$, so that the orbits of this action define an equivalence relation on the set of colorings. Show that if $\mathcal{G}$ is a square, and there are q different colors to choose from, then the number of inequivalent colorings of the vertices of $\mathcal{G}$ is $q(q+1)(q^2+q+2)/8$.

4. (a) Let p be a prime.

(i) Suppose G is a finite p-group acting on a set X. Show that $|\mathrm{Fix}\, G| \equiv |X| \bmod p$.

(ii) Using part (i), or otherwise, prove Cauchy's Theorem: if p divides the order of a finite group G, then G has an element of order p.

(b) State, but do not prove, the three parts (existence, uniqueness, arithmetic) of Sylow's Theorem.

(c) What is a *simple* group? Show there is no simple group of order (i) 36, (ii) 46, (iii) 56.