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OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2005

H. Diploma Arts
Master of Arts

MATHEMATICS

MA539 - ADVANCED ALGEBRA II

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Dr. J. Ward

Time allowed: **Two** hours.
Answer *three* questions.

1. (i) Explain how a field $\mathbf{K}$ may be viewed as a vector space over a sub–field $\mathbf{F}$ and hence define the degree $[\mathbf{K} : \mathbf{F}]$.
(ii) Define the term **splitting field** of a polynomial. State the roots of $x^4 + 1$ and calculate the degree of its splitting field over $\mathbf{Q}$.
(iii) Find the minimum polynomial of $i + \sqrt{2}$ over $\mathbf{Q}$.
(iv) Show that $x^4 + 1$ is *reducible* over $\mathbf{Q}(\sqrt{2})$, but does not split over $\mathbf{Q}(\sqrt{2})$. Deduce that $K = \mathbf{Q}(i + \sqrt{2})$ is the splitting field of $x^4 + 1$ over $\mathbf{Q}$.

p.t.o.

2. (i) What is meant by a "straight–edge and compass construction"?
(ii) State Gauss' Theorem concerning the values for which the regular n–gon can be constructed by straight edge and compass.
(iii) Using the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, or otherwise, **prove** that the angle $10°$ is not constructible.
(iv) Using part (ii) show that $18°$ **is** constructible. Deduce that the angle $n°$ is constructible $\Leftrightarrow 3|n$.
(v) Show that $\cos^{-1}\left(\frac{23}{27}\right)$ can be trisected using straight–edge and compass.

3. (i) Let p be a prime. Show that $x^p - 2$ is irreducible over $\mathbf{Q}$.
Prove that the splitting field of $x^p - 2$ over $\mathbf{Q}$ has degree $p(p-1)$.
(ii) Determine the Galois group G of $x^3 - 2$ over $\mathbf{Q}$ and establish that G is non–abelian of order 6.
(iii) Under the Galois correspondence find the (fixed) subfield corresponding to the subgroup of G of order 3.

4. (i) Let $\mathbf{GF}(q)$ be a finite field of order q ($= p^n$, p a prime $n \geq 1$). State the main properties of $\mathbf{GF}(q)$.
(ii) Prove that $\mathbf{GF}(q)$ is the splitting field of $x^{p^n} - x$ over $\mathbf{Z}_p$.
(iii) Let $f(x)$ be a monic irreducible polynomial of degree m over $\mathbf{Z}_p$. Prove that $f(x)$ divides $x^{p^n} - x \Longleftrightarrow m|n$. Hence or otherwise deduce that

$$p^n = \sum_{d|n} dN_p(d)$$

where $N_p(d)$ is the number of monic irreducible polynomials of degree d over $\mathbf{Z}_p$.
(iv) Calculate $N_2(1)$, $N_2(2)$, $N_2(4)$ and hence, or otherwise, factorise $x^{16} - x$ into irreducible factors over $\mathbf{Z}_2$.