1050

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

Semester II Examinations, 2004/2005

Exam Code(s)	2BS1, 2CS1, 2EL1, 2PT1, 3CS1.
Exam(s)	**Second Science and Arts**
Module Code(s)	MP208
Module(s)	Methods of Mathematical Physics(Honours)
Paper No.	
Special Paper	
External Examiner(s)	Professor B. Straughan;
Internal Examiner(s)	Professor M. S. Ó Confhaola; Dr. B. Gleeson; Dr. M. Meere; Dr. P. Ó Leary.
Instructions:	Attempt *THREE* questions, at least *ONE* from each section.
Duration	*TWO* hours
No. of Answer books	________
Requirements:	________
Handout	________
MCQ	________
Statistical Tables	YES, LOG TABLES
Graph Paper	________
Log Graph Paper	________
Other Material	________
No. of Pages	4 PAGES (Excluding Cover Page)
Department(s)	MATHEMATICAL PHYSICS

Section A

1. (a) Prove that the Laplace transform of the first derivative is given by

$$\mathrm{L}\left[\dot{f}(t)\right] = s\overline{f(s)} - f(0);$$

Hence prove that the Laplace transform of the second derivative is

$$\mathrm{L}\left[\ddot{f}(t)\right] = s^2\overline{f(s)} - sf(0) - \dot{f}(0).$$

(b) Use the Second Shift theorem to Laplace transform the function:

$$f(t) = \mathrm{H}(t-1)\sin(t).$$

(c) Solve the following set of simultaneous differential equations:

$$\frac{dx}{dt} = 2x(t) - 3y(t) \quad \text{AND} \quad \frac{dy}{dt} = y(t) - 2x(t),$$

subject to the following two initial conditions; $x(0) = 8$, $y(0) = 3$.

2. (a) Calculate the inverse Laplace transform below using, the method of partial fractions and the Convolution Theorem:

$$\mathrm{L}^{-1}\left[\frac{4}{s(s^2+4)}\right].$$

Verify your answers by Laplace transforming the resulting function.

(b) Use Laplace transforms to solve the following initial value problem:

$$\frac{d^2y}{dt^2} - 2a\frac{dy}{dt} + (a^2+b^2)\,y(t) = 0,$$

subject to the initial conditions $y(0) = 0$, $\dot{y}(0) = 1$, where $a, b > 0$.

3. (a) Find the differentiation, with respect to distance of the function

$$\phi(x,y,z) = 3x^2z + 3y - z^3,$$

along the parametric curve $\mathbf{r} = (u-1, u^2-1, 2-2u)$, at the origin.

(b) Prove that the point $(2,-1,2)$ lies on each of the surfaces

$$x^2 - 2y^2 + 2z^2 = 10 \quad \text{AND} \quad z = x^2 + y^2 - 3.$$

Prove that the two surfaces intersect at right angles to one another.

(c) If $\phi(x, y, z)$, $\mathbf{A}(x, y, z)$ are arbitary scalar and vector fields, prove

$$\operatorname{div}(\phi\mathbf{A}) = \operatorname{grad}\phi \cdot \mathbf{A} + \phi \operatorname{div}\mathbf{A}.$$

Hence prove that $\operatorname{div}(\mathbf{r}/r^3) = 0$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

4. Evaluate the line integral

$$\int_{(2,1,0)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r},$$

for the vector field

$$\mathbf{F} = (3x^2 - yz)\,\mathbf{i} + (3y^2 - xz)\,\mathbf{j} - xy\mathbf{k}$$

along each of the following paths:

(a) the straight line joining $(2, 1, 0)$ directly to $(1, 2, 3)$;

(b) the parametric curve $\mathbf{r}(u) = (2 - u, u^2 + 1, 3u^2)$;

(c) the successive straight line segments from $(2, 1, 0)$ to $(1, 1, 0)$, from $(1, 1, 0)$ to $(1, 2, 0)$ and from $(1, 2, 0)$ to $(1, 2, 3)$, in that order.

Can we deduce, based on our answers alone, whether the vector field $\mathbf{F}$ is conservative or not? Explain your answer.

Section B

5. The divergence theorem of Gauss can be written in the form

$$\iiint_{\mathrm{V}} \boldsymbol{\nabla} \cdot \mathbf{F}\, d\mathrm{V} = \iint_{\mathrm{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, d\mathrm{S}\,.$$

Explain what is meant by V, S and $\hat{\mathbf{n}}$ in the above equation. Verify the divergence theorem for the vector field

$$\mathbf{F} = (x + z)\,\mathbf{i} + (y + z)\,\mathbf{j} + (x + y)\,\mathbf{k},$$

where the region is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geqslant 0$.

6. Parabolic cylindrical coordinates (u, v, w) are related to rectangular Cartesian coordinates via

$$x = (u^2 - v^2)/2, \quad y = uv, \quad z = z,$$

where $-\infty < u < \infty$, $0 \leq v < \infty$, $-\infty < z < \infty$.

(a) Denoting the scale factors corresponding to (u, v, z) by (h_u, h_v, h_z), respectively, show that,

$$h_u = h_v = \sqrt{u^2 + v^2}, \quad h_z = 1.$$

(b) Find the coordinate vectors, $(\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_z)$, for parabolic cylindrical coordinates. Hence show that parabolic cylindrical coordinates are orthogonal.

(c) In the plane $z = 0$, plot the curves $u = 1$, $u = 2$, $v = 1$ and $v = 2$. Sketch in a typical $(\mathbf{e}_u, \mathbf{e}_v)$ pair at a point.

(d) Write down the line element and the volume element for parabolic cylindrical coordinates. Calculate the volume enclosed by the surfaces $u = 1$, $u = 2$, $v = 1$, $v = 2$, $z = 0$ and $z = 1$.

7. If A_i is a vector field in the x-frame where

$$A_1 = x_1^2, \quad A_2 = x_1 x_3, \quad A_3 = x_1 x_2,$$

and the transformatiom to the x'-frame (same origin) is

$$x_1' = \tfrac{1}{13}\left(5x_1 + 12x_2\right), \quad x_2' = \tfrac{1}{13}\left(12x_1 - 5x_2\right), \quad x_3' = x_3.$$

(a) Show that the transformation is orthogonal;

(b) Evaluate the components A_i' of the field in the new coordinates x_i';

(c) Verify that div $\mathbf{A}$ is invariant under this transformation.

TABLE OF LAPLACE TRANSFORMS

In all cases herein, a is a constant, and n is a positive integer:

$f(t) = \mathrm{L}^{-1}\left[\overline{f(s)}\right]$	$\overline{f(s)} = \mathrm{L}[f(t)]$
1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
t^n	$\dfrac{n!}{s^{n+1}}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$\mathrm{H}(t-a)$	$\dfrac{\exp[-as]}{s}$
$\delta(t-a)$	$\exp[-as]$

The Heaviside function, $\mathrm{H}(t-a)$, is defined by

$$\boxed{\mathrm{H}(t-a) = \begin{cases} 0 & \text{FOR} \quad 0 \leqslant t < a, \\ 1 & \text{FOR} \quad t \geqslant a. \end{cases}}$$

(In this context, the constant a is understood to be positive).