1052

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

Semester II Examinations, 2004/2005

Exam Code(s)	2CS1
Exam(s)	**Second Science**
Module Code(s)	MP214/215
Module(s)	Mathematical Physics(Honours)
Paper No.	SUMMER
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan;
Internal Examiner(s)	Professor M. S. Ó Confhaola; Dr. B. Gleeson.
<u>Instructions</u>:	Attempt *FIVE* questions, at least *TWO* from each section.
Duration	*THREE* hours
No. of Answer books	
<u>Requirements</u>:	
Handout	
MCQ	
Statistical Tables	YES, LOG TABLES
Graph Paper	
Log Graph Paper	
Other Material	
No. of Pages	4 PAGES (Excluding Front Page)
Department(s)	MATHEMATICAL PHYSICS

Section A

1. (a) A spinning top is precessing steadily, with an angular speed p, about a vertical axis through its end-point while it spins, with angular speed ω, about its own axis of symmetry. The following equation:

$$p = \frac{mg\bar{r}}{I\omega},$$

can be derived relating these two angular velocities, where m is the mass of the top, I is the moment of inertia of the top about its axis of symmetry, g is the acceleration due to gravity and $\bar{r}$ is the distance from the end-point of the top to the centre of mass. Use dimensional analysis to rewrite this equation in terms of dimensionless variables.

(b) A bead of mass $4m$ is threaded on a smooth horizontal straight wire. The bead is attached to a fixed point of the wire by a light (horizontal) spring with spring constant k. A particle of mass $8m$ is suspended from the bead by a heavy rigid rod of length $2a$ and mass $6m$. We can show that the relevant equations of motion are:

$$\begin{aligned} 18m\ddot{x} + 22ma\cos(\theta)\,\ddot{\theta} - 22ma\sin(\theta)\,\dot{\theta}^2 + kx &= 0, \\ 40ma^2\ddot{\theta} + 22ma\cos(\theta)\,\ddot{x} + 22mag\sin(\theta) &= 0. \end{aligned}$$

Rewrite the above equations in terms of dimensionless variables.

2. A thin hoop, of mass m and radius a, is placed on an inclined plane with an angle of inclination α to the horizontal. The coefficient of friction is:

$$\mu = \tfrac{2}{5}\tan\alpha.$$

Initially, the hoop has zero linear velocity, and has an angular velocity ω in the sense that the point of contact is slipping up the inclined plane.

(a) Show that at first the hoop will slide down the inclined plane, with constant linear acceleration $7g\sin\alpha/5$, for time interval $5a\omega/9g\sin\alpha$.

(b) Show that after this time, the hoop will again slide down the incline (in the opposite sense), with constant linear acceleration $3g\sin\alpha/5$.

3. A uniform rod of mass 6 M and length $2a$ is connected at one end to a smooth horizontal bar by a fixed hinge, and at the other end to a weight of mass M. The centre of the rod is suspended by a spring with

spring stiffness k, and natural length a, which is attached to the smooth horizontal wire above by a light bead. The whole system is constrained to move in a vertical plane. Show that the Lagrangian for this system is:

$$\mathrm{L} = 9\,\mathrm{M}\,a^2\dot{\theta}^2 - \tfrac{1}{2}ka^2\left(\cos\theta - 1\right)^2 + 8\,\mathrm{M}\,ga\cos\theta - 14\,\mathrm{M}\,ga,$$

where θ is the angle between the rod and the downward vertical, by finding first the Kinetic Energy and the Potential Energy of the system.

(a) Find the Lagrange equation of motion, for this system, for $\theta\,(t)$;

(b) Examine the potential energy of the system to find the equilibrium points of this system, and investigate their stability or instability.

4. Three springs, each of negligible mass and natural length a, are attached end-to-end and have a pair of particles, each of mass m, fixed to the points where they meet. The two outer springs have spring constant $3k$ while the middle spring has spring constant k. The system is stretched between two fixed points a distance $3a$ apart.

   You may assume that when the particles are constrained to move along the straight line of the springs, the KINETIC and POTENTIAL ENERGIES for the spring system, in terms of equilibrium coordinates x_1 and x_2, are:

$$\mathrm{T} = \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right),$$
$$\mathrm{V} = \tfrac{1}{2}k\left(3x_1^2 + 3x_2^2 + (x_2 - x_1)^2\right).$$

   Show that the natural frequencies for small longitudinal oscillations about the configuration of stable equilibrium, (that is, $x_1 = 0$ and $x_2 = 0$), are:

$$\sqrt{3k/m} \quad \text{AND} \quad \sqrt{5k/m}.$$

   Then show that a suitable set of natural coordinates for this problem is:

$$q_1 = x_1 + x_2, \quad \text{AND} \quad q_2 = x_1 - x_2.$$

   By substitution of the above natural coordinates in the Lagrangian, show that these natural coordinates decouple Lagrange's equations of motion.

5. (a) It can be shown that the Lagrangian of a mass m, being subject to a system of conservative forces, in *plane-polar coordinates* (r, θ) is:

$$\mathrm{L} = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \mathrm{V}\,(r),$$

   where the potential energy is a function of r. Find the Hamiltonian for this system, and hence find the Hamiltonian equations of motion.

(b) In Hamiltonian mechanics, what is meant by the term *ignorable coordinates*, and what are the ignorable coordinates of this system?

(c) State the one-dimensional Time Dependent Schrödinger Equation.

(d) Use a separated variables solution, $\Psi(x,t) = \psi(x)\,\phi(t)$, to show the function $\psi(x)$ satisfies the Time Independent Schrödinger Equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \mathrm{V}(x)\,\psi(x) = E\psi(x),$$

where the constant E is the constant of separation. Find $\phi(t)$ also.

Section B

6. (a) Suppose a special breed of cat, which lives for exactly *seven* years according to his own body clock, is placed in a spaceship when it is born, and send to Alpha Centauri at speed v. How far from Earth, relative to Earth's frame of reference will the cat be when it dies, if

$$\text{i)}\quad v = 3\,\mathrm{c}/5, \quad \text{ii)}\quad v = 5\,\mathrm{c}/13, \quad \text{iii)}\quad v = 12\,\mathrm{c}/13.$$

(b) A 2 m rod is in a satellite moving at speed $\frac{1}{2}$ c with respect to Earth. What length is this rod according to an observer in the satellite?

(c) Show that if a particle of mass m moves along the $\bar{x}$ axis of frame $\overline{\mathrm{S}}$ with speed $\bar{u}$ then it moves along the x axis of S with speed u where

$$u = \frac{\bar{u} + v}{1 + \bar{u}v/c^2},$$

where v is the speed of $\overline{\mathrm{S}}$ along the x axis in standard configuration.

(d) Two particles are observed moving towards each other, each with speed 90% c, with respect to the lab. What is their relative speed?

7. (a) Find the general solution to Bessel's equation, of order 1/2:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \tfrac{1}{4}\right)y(x) = 0,$$

using the Method of Frobenius. No other method is acceptable.

(b) Use the transformation $y(x) = x^2 u(x)$ in the differential equation

$$x^2y'' - 4xy' + 2\left(2x^2 + 3\right)y = 0.$$

Hence, or otherwise, write down the general solution for $y(x)$.

8. Use a separable variables solution to solve the differential equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} : \quad 0 < x < L, \quad t > 0,$$

where α is a real positive constant, subject to boundary conditions

$$u_x(0,t) = u_x(L,t) = 0, \quad t \geqslant 0.$$

and the initial condition $u(x,0) = f(x)$ on the interval $0 \leqslant x \leqslant L$.

9. Use a separable variables solution to solve the differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 : \quad 0 < \theta < \alpha, \quad a < r < b,$$

where $\alpha < 2\pi$ is a real positive constant, subject to the conditions

$$u(r,0) = u(r,\alpha) = 0, \quad a < r < b,$$

on the straight edges $\theta = 0$, $\theta = \alpha$, and the boundary conditions

$$u(a,\theta) = 0 \quad u(b,\theta) = f(\theta), \quad 0 < \theta < \alpha.$$

on the curved edges $r = a$, $r = b$, where a, b are both positive.

10. (a) Let $u(x,t)$ be the solution of the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) : \quad 0 < x < L, \quad t > 0,$$

where $k(x) > 0$, subject to the initial/boundary condition(s)

$$u(0,t) = u_0, \quad u_x(L,t) = 0, \quad t \geqslant 0,$$
$$u(x,0) = g(x), \quad 0 \leqslant x \leqslant L.$$

Use the following cross-sectional measure of $u(x,t)$:

$$F(t) = \int_0^L u^2(x,t)\,dx,$$

to determine whether or not this Initial Boundary Value Problem is well-posed (assuming that the solution $u(x,t)$ exists without proof).

(b) Consider this BVP: Suppose $u(x,t)$ satisfies the Wave Equation

$$u_{tt} - c^2 u_{xx} = 0 : \quad 0 < x < L, \quad t > 0,$$

subject to the boundary conditions

$$u_x(0,t) = 0, \quad u(L,t) = 0, \quad t \geqslant 0.$$

Establish a '*conservation law*' for this Boundary Value Problem.