1053

Ollscoil na hÉireann, Gaillimh

National University of Ireland, Galway

Semester Two Examinations, 2004/2005

Exam Code(s)	2BE1, 2BG1, 2BI1, 2BM1, 2BN1, 2BP1, 2BV1
Exam(s)	Second Engineering
Module Code(s)	MP250
Module(s)	Mathematical Physics
Paper No	2
Repeat Paper	
Special Paper	
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Internal Examiner(s)	Dr. M. Ó Confhaola
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Instructions:

Answer { THREE questions from Section A and TWO questions from Section B

All questions carry the same marks.

A compendium of useful formulae is attached.

Duration	3 hours
No. of Answer books	1

Requirements

Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	

No. of Pages	4
Department(s)	Mathematical Physics

Section A

1. The gates of a canal lock extend the full height of the lock. When a vessel is being lowered in the lock the gates at the outlet are opened at the rate of 0.3 m/min. The currents produced by the outflow of water cause the vessel to pull at its moorings.

   The tension $\widehat{T}$ in the hawsers depends on the speed V of opening, and the height h of the gates, the density ρ and viscosity μ of the water, and the acceleration due to gravity g.

   Use dimensional analysis to find how $\widehat{T}$ depends on the other variables.

   In a $\frac{1}{36}$th scale geometrically similar model of the system, the maximum tension in the hawsers is 40 N when the gates are opened at the proper rate. What is the correct rate of opening of the gates to ensure kinematic similarity? Obtain an estimate for the maximum tension in the hawsers of the prototype if you can assume that viscous effects are unimportant.

2. A plane compound pendulum is formed out of a uniform circular disk of mass M and radius a and a thin uniform rod AB of mass $\frac{1}{4}M$ and length $7a$. The end B of the rod is rigidly attached to a point of the edge of the disk - the rod being normal to the circular edge of the disk. The other end of the rod is fixed in position and the rod is allowed to oscillate freely in the vertical plane containing the rod and the disk.

   (a) Deduce the equation of motion of the compound pendulum from the principle of angular momentum and, hence, identify the length of the equivalent simple pendulum.

   (b) The pendulum is initially held with the rod hanging vertically downwards. What angular velocity must the pendulum be given to ensure that it completes at least one revolution?

3. A uniform solid spherical ball, of mass m and radius a and moment of inertia $\frac{2}{5}ma^2$ about a diameter, and a uniform thin spherical shell, also of mass m and radius a but of moment of inertia $\frac{2}{3}ma^2$ about a diameter, are both released from rest at the top of an inclined plane. Each moves down the incline along a line of greatest slope.

   If the coefficient of friction is

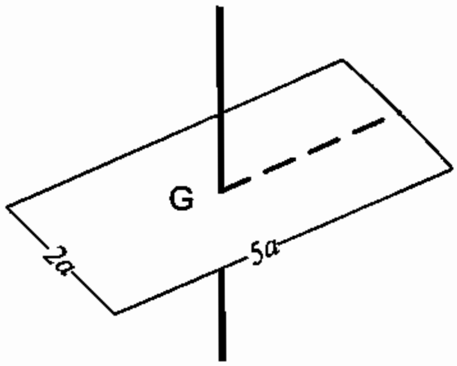
   $$\mu = \frac{1}{3}\tan\alpha\ ,$$

   where α is the angle of inclination of the plane, show that one of the two balls will roll while the other will slide. Identify clearly which rolls and which slides. Which one reaches the bottom of the incline first?

4. A light string passes over a pulley of mass $3M$ which is fixed in position but free to rotate. It carries a mass $12M$ at one end, the other end being attached to another pulley of mass $6M$ over which passes a second light string whose ends carry masses $2M$ and $4M$. The pulleys (treated as uniform disks of radius a) are rough enough to prevent the strings slipping.

   Obtain Lagrange's equations of motion for the system and use these equations to find the linear accelerations of the $12M$, $2M$ and $4M$ masses and the pulley of mass $6M$.

5. A uniform rectangular plate, of sides $2a$ and $5a$ and mass m, is spinning steadily about a *vertical* shaft through its centre of mass G as shown in the accompanying diagram. The vertical axis of rotation lies in a plane perpendicular to the plate which bisects the plate along a line through G parallel to the side of length $5a$. The shaft makes an angle $\tan^{-1}(2/3)$ with this line through G.
Determine *(i)* the angular momentum of the plate relative to G, $\mathbf{h}_G$, *(ii)* the moment of inertia of the plate about the axis of rotation, *(iii)* the couple that must act on the plate to maintain this motion, and *(iv)* the reaction torque exerted by the plate on the shaft.



Section B

6. Answer each of the following two parts:

(a) The curve C is described by the parametric equation

$$\mathbf{r} = (t^2+1)\,\mathbf{i} + (4t-3)\,\mathbf{j} + (2t^2-6t)\,\mathbf{k}$$

and passes through the point $(5, 5, -4)$. Find the unit tangent vector to the curve C at this point. Hence, or otherwise, find the directional derivative in the direction of the curve of the scalar field

$$\phi = x^2yz + 4xz^2$$

at the point $(5, 5, -4)$.

(b) Verify, by calculating each term separately, the identity

$$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$$

for the scalar field $\phi = x^2yz$ and the vector field $\mathbf{F} = 2xz^2\mathbf{i} - yz\,\mathbf{j} + 3xz^3\mathbf{k}$.

7. Verify Green's Theorem in the plane for the line integral

$$\oint \left[(2xy - x^2)\,dx + (x + y^2)\,dy\right]$$

around the boundary of the finite region between the two parabolas $y = x^2$ and $x = y^2$.

8. Verify the divergence theorem of Gauss for the vector field

$$\mathbf{V} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$$

when the volume of integration is the cylindrical region $x^2 + y^2 \leq 4, \quad 0 \leq z \leq 3$.

Useful Formulae

Dimensional Analysis

The coefficient of vscosity μ is defined by a dimensionally homogeneous equation of the form

$$\text{force per unit area} \quad = \quad \mu \cdot (\text{gradient of a velocity})$$

Rigid Body Dynamics (for a rigid body free to move with one point fixed)
Principle of Angular Momentum:

$$\left(\frac{d}{dt}\mathbf{h}_O\right)' + \boldsymbol{\omega} \times \mathbf{h}_O = \mathbf{C}_O$$

Principal Moments of Inertia of a uniform rectangular plate:

If the edges of the plate are $2a$ and $2b$ and the mass of the plate is m, then the principal moments of inertia at the centre of mass of the plate are $\frac{1}{3}\,mb^2$, $\frac{1}{3}\,ma^2$ and $\frac{1}{3}\,m\left(a^2+b^2\right)$, the corresponding principal axes being parallel to the sides of length $2a$ and $2b$, in turn, and perpendicular to the plate.

Lagrange's Equations for a Simple Lagrangian System

For a system with n degrees of freedom and corresponding coordinates $q_i,\ i = 1, 2, ..., n$, the Lagrangian function is given by

$$L\left(q, \dot{q}\right) = T\left(q, \dot{q}\right) - V\left(q\right).$$

The equations of motion are then given by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, ..., n.$$

Vector Differential Calculus

$$\boldsymbol{\nabla} = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

Green's Theorem in the plane

$$\oint\limits_C \left[f\left(x, y\right) dy - g\left(x, y\right) dx\right] = \iint\limits_R \left(\frac{\partial f}{dx} + \frac{\partial g}{\partial y}\right) dA$$

where C is the closed boundary of the planar region R and f and g are differentiable functions defined in R and on C.

The divergence theorem of Gauss

$$\iiint\limits_V \boldsymbol{\nabla} \cdot \mathbf{F}\; dV = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\; dS\,.$$

where the volume V is bounded by the closed surface S, the differentiable vector field $\mathbf{F}$ is defined throughout V and on S and $\mathbf{n}$ is a unit *outward* normal vector on S.