10541

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

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Exam Code(s)	3BS3;3BS9;3EL1;3PT1;3PT2
Exam(s)	Third Science
Module Code(s)	MP303
Module(s)	Quantum Mechanics (Pass)
Paper No	1
Repeat Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Micheál Ó Confhaola

Instructions: Full marks for **five** questions

Duration	3hrs
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	4
Department(s)	Mathematical Physics

1. A quantum particle is confined to a one dimensional infinite square well potential

$$V(x) = \begin{cases} 0 & 0 \le x \le a, \\ \infty & otherwise \end{cases}$$

Show that the allowed energy levels of the particle and the corresponding normalized eigenfunctions are given by

$$E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2}, \quad \psi_n(x) = \sqrt{\frac{2}{a}}\,\sin(\frac{n\pi x}{a})$$

where n is an integer taking the values $1, 2, 3.....$

Consider a particle in an infinate square well potential whose wave function at $t = 0$ is

$$\Psi(x,0) = \frac{3}{\sqrt{30}}\psi_1(x) + \frac{4}{\sqrt{30}}\psi_2(x) + \frac{1}{\sqrt{6}}\psi_4(x)$$

(a)Find the average energy of the particle.

(b)Find the state $\Psi(x,t)$ at a later time t and the average value of the energy. Compare the result with the value obtained in (a)

2. A uniform stream of particles of mass m and energy $E > 0$ which move in the positive $x - direction$ impinges normally on the potential barier

$$V(x) = \begin{cases} 0, & \text{for } x < 0 \\ -V_0, & \text{for } x \ge 0 \end{cases}$$

(a)Find the reflection coefficient R.

(b)If $E = \frac{V_0}{3}$, show that $R = \frac{1}{9}$

3. (a) Explain what is meant by an Hermitian operator and prove explicitly that $p_x = -i\hbar\frac{\partial}{\partial x}$ is an Hermitian operator on the space of square integrable functions.
(b) Show that the eigenvalues of an Hermitian operator are real and that the eigenfunctions corresponding to the distinct eigenvalues are orthogonal.
(c) If A and B are Hermitian operators which commute and if Ψ_1 and Ψ_2 are two eigenfunctions of A with different eigenvalues show that (i) $(\Psi_1, B\Psi_2) = 0$ and (ii) $B\Psi_1$ is also an eigenvalue of A belonging to the same eigenvalue.

4. Consider a particle described by the wavefunction

$$\Psi(x,t) = Ae^{-|x|/L}$$

(a) Find A so that $\Psi(x)$ is properly normalized.

(b) What is the probability of finding the particle in the region $[-L, L/2]$?

(e) Find $\overline{\Psi(p)}$, i.e the momentum space wave function.

5. The Hamiltonian operator for a one-dimensional harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

where $[x,p] = i\hbar$.
By transforming to operators a and $a^\dagger$ where

$$a = \frac{1}{\sqrt{2}}\left(\frac{ip}{\sqrt{\hbar m\omega}} + \sqrt{\frac{m\omega}{\hbar}}x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{ip}{\sqrt{\hbar m\omega}} + \sqrt{\frac{m\omega}{\hbar}}x\right),$$

show that

$$H = (a^\dagger a + \frac{1}{2})\hbar\omega = (aa^\dagger - \frac{1}{2})\hbar\omega, \ [a,H] = a\hbar\omega, [a^\dagger,H] = -a^\dagger\hbar\omega$$

Hence or otherwise, show that the eigenvalue spectrum of H is

$$E_n = (n + \frac{1}{2})\hbar\omega; \ n = 0,1,2,..$$

6. If A and B are two Hermitian operators and $[A,B] = i\hbar$, derive the uncertainty relation

$$\Delta A \Delta B \geqslant \frac{\hbar}{2}$$

where $(\Delta A)^2 = \left\langle (A - \langle A \rangle)^2 \right\rangle$
Prove that $\Delta A = 0$ if and only if Ψ is an eigenstate of A.

7. The components of the angular momentum $\mathbf{J}$ of a system satisfy the commutation relations

$$J_\alpha J_\beta - J_\beta J_\alpha = i\hbar J_\gamma$$

where (α, β, γ) is any cyclic permutation of (x,y,z). If the operators J_+ and J_- are defined by

$$J_+ = J_x + iJ_y, \quad J_- = J_x - iJ_y$$

show that

$$[J_z, J_+] = \hbar J_+ \ , \ [J_z, J_-] = -\hbar J_-$$

$$J_+J_- = J^2 - J_z^2 + \hbar J_z \ , \ J_-J_+ = J^2 - J_z^2 - \hbar J_z$$

Outline the steps by which it is established that the simultaneous eigenstates of J^2 and J_z are written in the form $|jm\rangle$ where

$$J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle, \quad J_z|jm\rangle = m\hbar|jm\rangle$$

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$.......and $m = -j, -j+1,$.........$j-1, j$.

8. Show that the time derivative of the expectation value of any observable A is given by

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle [A,H]\rangle + \left\langle \frac{\partial A}{\partial t}\right\rangle$$

If A, B, C are any three operators, show that

$$[A,BC] = [A,B]C + B[A,C], \quad [AB,C] = A[B,C] + [A,C]B$$

Prove the following results for a *free* particle moving in one dimension.

$$\frac{d}{dt}\langle p\rangle = 0$$

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle$$

$$\frac{d}{dt}\langle x^2\rangle = \frac{1}{m}\langle xp + px\rangle$$

9. A particle of mass m is confined within a sphere of radius a by the spherically symmetric infinite square well potential

$$V(r) = \begin{cases} 0 & , r \le a \\ \infty & , r > a \end{cases}$$

What boundary condition must the wave function satisfy on the sphere $r = a$?. Show that the stationary states with spherically symmetric wave functions(i.e. s-states) are of the form

$$\psi_n(r) = \frac{1}{\sqrt{2\pi a}}\frac{1}{r}\sin\left(\frac{n\pi r}{a}\right)$$

for $r < a$, where n is a positive integer. If the particle is in a state described by the (normalized) wave function

$$\psi(r) = \sqrt{\frac{15}{2\pi a^5}}\,(a-r)$$

show that the expectation value of the energy is $\frac{5\hbar^2}{ma^2}$

[Note $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$]