1055

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

Semester II Examinations, 2004/2005

Exam Code(s)	3BS3,3BS5,3EL1,3EL2,3PT1,3PT2
	4BS2,4BS3,4BS4,4BS9,4CS2,1MF2,1SD1,1EM2
Exam(s)	Third Arts, Third Science, Fourth Science, H.Dip.Appl.Sc.,M.Sc.
Module Code(s)	MP307
Module(s)	Modelling II
Paper No	1
Repeat Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Mícheál Ó Confhaola
	Dr. Michael Tuite

Instructions: Full marks for **THREE** correctly anwered questions.

Duration	2hrs
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	

1.

(a) Describe what is meant by a finite ergodic Markov process with transition probability matrix P. Define the equilibrium probabilities π_i for a finite ergodic system and show that they obey the matrix equation $\Pi = \Pi P$ where Π is a row matrix with elements π_i. Describe a sufficient condition for P which guarantees ergodicity.

(b) Describe an infinite queue with a nearest-neighbour Markov process where the arrival and servicing patterns are independent of the queue size. Find under what conditions this system is ergodic and find the equilibrium probabilities.

(c) Customer arrivals in a queue are described by a Poisson process with parameter α. Show that the probability density function for the waiting time t for a customer to arrive is given by the exponential distribution $f(t) = \alpha\exp(-\alpha t)$. Show that the average time taken for 1 customer to arrive is $1/\alpha$.

(d) On average, an airport with a single runway allows one landing every 4 minutes with 12 aircraft arriving per hour. Using a Poisson model for the arrival and servicing mechanism, find the equilibrium probability that more than 5 aircraft are waiting to land.

2. A queue of maximum size r has a Poisson arrival pattern with parameter α_k and Poisson servicing pattern with parameter β_k where k is the queue size with $0 \le k \le r$.

(a) Explain why $\alpha_r = \beta_0 = 0$. Assuming that the system is ergodic, show that the equilibrium probabilities are given by

$$\pi_k = \rho_k \rho_{k-1} \ldots \rho_1 \pi_0$$

for $0 < k \le r$ where $\rho_k = \alpha_{k-1}/\beta_k$.

(b) Consider an infinite queue with ample servers where the arrival pattern is independent of the queue size whereas the probability of a servicing is proportional to the queue size. Show that the equilibrium probabilities are given by a Poisson distribution.

(c) A telephone enquiry service consists of many operators each of whom can deal with 1 customer enquiry per minute on average. If 600 calls per hour arrive on average what is the equilibrium probability distribution? Find the average number of customers being serviced.

3.

(a) Consider the notorious IBM RANDU multiplicative congruential random number generator

$$I_{k+1} = aI_k \bmod m$$

for $k = 0, 1, \ldots$ and with $m = 2^{31}$ for $a = 2^{16} + 3$ (which has period of 2^{16}). Show that all points (x_k, x_{k+1}, x_{k+2}) where $x_k = I_k/m$ lie on one of 15 parallel planes within the unit cube.

(b) Assuming that pseudo-random numbers can be uniformly generated on $[0, 1]$, describe algorithms which simulate

  (i) an exponential decay process with probability density function $f(t) = \lambda\exp(-\lambda t)$ where $\lambda > 0$,

  (ii) a normal distribution with mean 0 and standard deviation 1.

(c) Consider a finite ergodic Markov process with transition matrix p_{ij} and equilibrium probabilities π_i. Show that if for some numbers a_i the Detailed Balance condition

$$a_i p_{ij} = a_j p_{ji},$$

is satisfied (i,j not summed) then $a_i = A\pi_i$ for an appropriate constant A.

(d) Hence describe the Metropolis algorithm as a Markov process which can simulate any given finite probability distribution π_i. Assuming that the algorithm is ergodic, show that the Detailed Balance condition $\pi_i p_{ij} = \pi_j p_{ji}$ is satisfied and hence the equilibrium probability distribution is π_i.

4. Consider a species in which no individuals live beyond 3 years. Divide the population into four age groups labelled by $n = 0, 1, 2$ and 3. Assume that only the second and third groups can reproduce. For each group n let b_n denote the birth rate and d_n the death rate. Find a matrix **A** such that $\mathbf{P}(t+1) = \mathbf{A}\mathbf{P}(t)$ with

$$\mathbf{P} = \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

where $P_n(t)$ denotes the population size of age group n at time t.

(a) Show that the eigenvalues of **A** satisfy the equation

$$\lambda^4 - b_1(1-d_0)\lambda^2 - b_2(1-d_0)(1-d_1)\lambda = 0.$$

(b) If $d_0 = 0.2, d_1 = 0.2,\ d_2 = 0.6$, and $b_1 = 0.5,\ b_2 = 1.0$, show that there is one eigenvalue greater than 1. What is the significance of this?

(c) Considering the evolution of the species over three years, where the initial population is $P_0 = 2, P_1 = P_2 = 1$ and $P_3 = 0$.

5. Two competing species with population sizes P_1, P_2 evolve according to the following differential equations

$$\frac{dP_1}{dt} = 10P_1 - 0.2P_1^2 - 0.2P_1P_2,$$
$$\frac{dP_2}{dt} = 4P_2 - 0.1P_1P_2 - 0.1P_2^2.$$

(a) Give an interpretation for the various terms in these equations.

(b) Find the equilibrium points for P_1, P_2.

(c) Show that for any initial non-zero populations then $P_1(t) \to 50$ and $P_2(t) \to 0$ as $t \to \infty$. What is the interpretation of this result?