1056

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

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Exam(s)	Third and Fourth Science
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Module(s)	Quantum Mechanics (Honours)
Paper No	1
Repeat Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Mícheál Ó Confhaola

Instructions: Full marks for **FIVE** questions

Duration	3hrs
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	5
Department(s)	Mathematical Physics

1. Consider a particle in a potential

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & otherwise \end{cases}$$

Show that the allowed energy levels of the particle are given by

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

and find the corresponding normalized energy eigenfunctions.

The state of the particle at time $t = 0$ is

$$\Psi(x,0) = A\left(\frac{1}{\sqrt{19}}\psi_1(x) + \frac{2}{\sqrt{19}}\psi_2(x) + \sqrt{\frac{2}{19}}\,\psi_3(x) + \sqrt{\frac{3}{19}}\,\psi_4(x) \right)$$

(a) Determine the normalization constant A

(b) What is the probability, when the energy of the particle in the state $\Psi(x,0)$ is measured, of finding a value smaller than $\frac{3\hbar^2\pi^2}{ma^2}$?

(c) What is the mean value and the root-mean-square deviation of the energy of the particle in the state $\Psi(x,0)$?

(d) Calculate the state vector $\Psi(x,t)$ at the instant t. Do the results found in (b) and (c) at the instant $t = 0$ remain valid at an arbitary time t?

(e) When the energy is measured, the result $\frac{8\hbar^2\pi^2}{ma^2}$ is found. After measurement what is the state of the system? What is the result if the energy is measured again?

2. Use the Schrodinger equation for a particle moving in a potential $V(\mathbf{r})$

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(\mathbf{r})\Psi$$

to derive the equation

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + div\ \mathbf{j}(\mathbf{r},t) = 0$$

where

$$\rho(\mathbf{r},t) = |\Psi(\mathbf{r},t)|^2, \quad \mathbf{j}(\mathbf{r},t) = \frac{\hbar}{2mi}[\Psi^*(\nabla\Psi) - \Psi(\nabla\Psi^*)]$$

Interpret $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$.

A uniform stream of particles of mass m and energy $E = \frac{V_0}{2}$ which move in the positive x –direction impinges normally on the potential barrier

$$V(x) = \begin{cases} 0, & x \le 0 \\ V_0, & 0 < x \le a \\ 0, & x > a \end{cases}$$

Show that the transmission coefficient T is given by $T = \cosh^{-2}(ka)$, where $k^2 = \frac{mV_0}{\hbar^2}$.

[Note: $div(\phi\mathbf{A}) = \phi div\mathbf{A} + (\nabla\phi)\cdot\mathbf{A}$]

3. The Hamiltonian operator for a one-dimensional harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

where $[x,p] = i\hbar$.
By transforming to operators a and $a^\dagger$ where

$$a = \frac{1}{\sqrt{2}}\left(\frac{ip}{\sqrt{\hbar m\omega}} + \sqrt{\frac{m\omega}{\hbar}}x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{ip}{\sqrt{\hbar m\omega}} + \sqrt{\frac{m\omega}{\hbar}}x\right),$$

show that

$$H = (a^\dagger a + \frac{1}{2})\hbar\omega = (aa^\dagger - \frac{1}{2})\hbar\omega, \ [a,H] = a\hbar\omega, [a^\dagger,H] = -a^\dagger\hbar\omega$$

Hence or otherwise, show that the eigenvalue spectrum of H is

$$E_n = (n + \frac{1}{2})\hbar\omega; \ n = 0,1,2,..$$

Show that if $H = T + V$ is the energy operator for the one-dimensional harmonic oscillator, then $\left[H, \frac{i}{2\hbar}xp\right] = T - V$, where T and V are the kinetic and potential operators respectively. By taking the expectation value of this relation in an energy eigenstate, prove that in such a state the expectation values of the kinetic and potential energies are equal.
[Note: If A, B, C are operators, then $[A,BC] = [A,B]C + B[A,C]$: $[AB,C] = A[B,C] + [A,C]B$]

4. (a) If A and B are two Hermitian operators and $[A,B] = i\hbar$, derive the uncertainty relation

$$\Delta A \Delta B \geqslant \frac{\hbar}{2}$$

where $(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle$
Prove that $\Delta A = 0$ if and only if Ψ is an eigenstate of A.
(b) Show that the time derivative of the expectation value of any observable A is given by

$$\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar}\langle [A,H] \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle$$

where H is the Hamiltonian operator
(c) If A is an operator such that $[A,H] = 0, \frac{\partial A}{\partial t} = 0$, show that ΔA is constant in time.

5. A quantum system can exist in two states $|a_o\rangle$ and $|a_1\rangle$ which are normalized eigenstates of the observable A with eigenvalues 0 and 1 respectively. The Hamiltonian operator H is defined by

$$\widehat{H}|a_o\rangle = \alpha|a_o\rangle + \beta|a_1\rangle, \ \widehat{H}|a_1\rangle = \beta|a_o\rangle + \alpha|a_1\rangle$$

where α and β are real. If the system is in the state $|a_o\rangle$ at time $t = 0$, show that at time t its state is

$$|\Psi(t)\rangle = e^{-i\alpha t/\hbar}\left[\cos\left(\frac{\beta t}{\hbar}\right)|a_o\rangle - i\sin\left(\frac{\beta t}{\hbar}\right)|a_1\rangle\right]$$

An observable A is measured at time $t = T$, but the value is lost. It is measured again at time $t = 2T$. Find the probability that the second measurement of A gives the result 0.

6. The three components of spin of an electron satisfy the relations

$$[S_x,S_y] = i\hbar S_z,\ [S_y,S_z] = i\hbar S_x,\ [S_z,S_x] = i\hbar S_y,\ S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}\hbar^2$$

Prove that $S_+ = S_x + iS_y$, and $S_+ = S_x - iS_y$ satisfy

$$[S_z,S_+] = \hbar S_+,\ [S_z,S_-] = -\hbar S_-,\ S_+S_- = \frac{3}{4}\hbar^2 - S_z^2 + \hbar S_z$$

Deduce that if $|+\rangle$ is a normalized eigenvector of S_z with eigenvalue $\frac{\hbar}{2}$, then $|-\rangle = \frac{1}{\hbar}S_-|+\rangle$ is a normalized eigenvector of S_z with eigenvalue $-\frac{\hbar}{2}$ satisfying $S_+|-\rangle = \hbar|+\rangle$

Show that in the $\{|+\rangle,|-\rangle\}$ basis the matrix representing S_x is $\frac{\hbar}{2}\sigma_x$ where $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and obtain the corresponding matrices representing S_y and S_z. If α is real, find a linear combination of the vectors $|+\rangle$ and $|-\rangle$ that is a normalized eigenvector of $S_x\cos\alpha + S_y\sin\alpha$ with eigenvalue $\frac{\hbar}{2}$

7. Discuss the vector addition of two angular momentum operators $\mathbf{J}_1$ and $\mathbf{J}_2$ each of which satisfy the commutation relations for angular momentum. If $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ show that the components of $\mathbf{J}$ satisfy the commutation relations for angular momentum which may be written

$$\mathbf{J} \times \mathbf{J} = i\hbar\mathbf{J}$$

Let $|JM\rangle$ be the common eigenstates of $\{J_1^2, J_2^2, J^2, J_z\}$ and are written as

$$|JM\rangle = \sum_{\substack{m_1,m_2 \\ m_1+m_2=M}} \langle j_1m_1j_2m_2|JM\rangle|j_1m_1j_2m_2\rangle$$

where $|j_1m_1j_2m_2\rangle$ is an eigenstate of $J_1^2, J_{1z}, J_2^2, J_{2z}$.
For the case $j_1 = 1,\ j_2 = 1$ evaluate the coefficients $\langle 1010|20\rangle$ and $\langle 1011|11\rangle$
[Note: $J_\pm|JM\rangle = \hbar\sqrt{J(J+1) - M(M\pm 1)}\,|JM\pm 1\rangle$]

8. The Hamiltonian operator in spherical polar co-ordinates is

$$H = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + V(r,\theta,\phi)$$

where m is the mass of the particle.
If the potential V is independent of θ and ϕ, explain why the energy eigenfunctions can also be chosen to be eigenfunctions of the total orbital angular momentum operator L^2.
A particle of mass m is confined within a sphere of radius a by the spherically symmetric square well potential

$$V(r) = \begin{cases} -V_0, & \text{for } r < a \\ 0, & \text{for } r > a \end{cases}$$

Show that the energies of the spherically symmetric ($\ell = 0$) states are determined by the condition

$$ka\cot(ka) = -\beta a,\ \text{where } k^2 = \frac{2m(V_0 - |E|)}{\hbar^2},\ \ \beta^2 = \frac{2m|E|}{\hbar^2}$$

Show that no bound state of such a system exists unless $V_0 \geq \frac{\hbar^2\pi^2}{8ma^2}$

9. The expression

$$E_\psi = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

defines the average value of E for a state ψ. Show that the stationary values of E_ψ with respect to arbitary variation of ψ are the eigenvalues of H and that if ψ_0 corresponds to the lowest value E_0,

$$E_\psi \geq E_0$$

Use the above method to estimate the ground state energy of a particle in the potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ cx, & x > 0 \end{cases}$$

Use the trial function $\psi(x) = xe^{-\alpha x}$

$\left[Note: \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}\right]$