1057

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

Summer Examinations, 2004/2005

Exam Code(s)	3BS4,3BS5
Exam(s)	Third Science
Module Code(s)	MP329
Module(s)	Calculus of Variations and Classical Elasticity
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Professor Micheál Ó Confhaola
	Dr. Pat O' Leary
	Dr. M.G. Meere
Instructions:	Attempt **FIVE** questions
Duration	**THREE HOURS**
No. of Answer books	1
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	3 (excluding this page)
Department(s)	Mathematical Physics

1. Consider the functional

$$J(y) = \int_0^{x_2} y^{1/2}(1 + (y')^2)^{1/2}dx$$

with $y \in C^2[0,x_2]$ such that $y(0) = y_1, y(x_2) = y_2, x_2, y_1, y_2$ being fixed positive constants. Prove that the curves of the foregoing type giving the functional a stationary value are of the parabolic type

$$2(cy - 1)^{1/2} = cx + d$$

where c, d are suitable constants, and that c is determined by

$$2\{(cy_2 - 1)^{1/2} - (cy_1 - 1)^{1/2}\} = cx_2.$$

Prove that there are two such curves, one, or none depending on whether

$$x_2^2 < 4y_1y_2, x_2^2 = 4y_1y_2, x_2^2 > 4y_1y_2,$$

respectively.

Hint: The Euler-Lagrange equation for a functional of the form

$$J(y) = \int_0^{x_2} F(y,y')dx$$

has first integral

$$F - y'\frac{\partial F}{\partial y'} = \text{constant.}$$

2. a. Find the extremal for the following functional satisfying the given boundary conditions:

$$J(y) = \int_0^1 (1 + (y'')^2)dx,\ y \in C^4[0,1],\ y(0) = 0,\ y'(0) = 1,\ y(1) = 1,\ y'(1) = 1.$$

b. Find the extremal for

$$J(y) = \int_1^e (x^2(y')^2/2 - y^2/8)dx \text{ for } y \in C^2[1,e] \text{ and } y(1) = 1, y(e) \text{ unspecified.}$$

3. a. Consider a quadratic functional in two independent variables (x,y) of the form

$$J(z) = \int_D \left\{\frac{1}{2}T(x,y)(z_x^2 + z_y^2) - w(x,y)z\right\}dA,$$

where $z = z(x,y)$, and D is some closed domain in the plane with sufficiently smooth boundary C. Here $T \in C^1(D \cup C)$, $T > 0$ in $D \cup C$, $w \in C(D \cup C)$, and $z \in C^2(D \cup C)$. Also, z satisfies the boundary condition $z = 0$ on C.

Show that the problem for stationary values for $J(z)$ leads to the boundary value problem:

$$(Tz_x)_x + (Tz_y)_y = -w \text{ in } D,$$
$$z = 0 \text{ on } C.$$

Show that the extremal arising furnishes an absolute minimum.

b. Consider the following boundary value problem in a square region for the function $z = z(x,y)$:

$$\nabla^2 z = -1 \text{ in } -1 < x < 1,\ -1 < y < 1,$$
$$z = 0 \text{ on the boundary of the square.}$$

Obtain the best approximation to z of the type

$$Z = c(1 - x^2)(1 - y^2),$$

by seeking the minimum of the appropriate integral (see part **a.** of this question).

4. a. Derive a necessary condition for an extremum for the following isoperimetric problem. Minimize $(y_1(x), y_2(x) \in C^2[a,b])$

$$J(y_1, y_2) = \int_a^b F(x, y_1, y_2, y_1', y_2')dx$$

subject to

$$\int_a^b G(x, y_1, y_2, y_1', y_2')dx = C$$

and

$$y_1(a) = A_1, y_2(a) = A_2, y_1(b) = B_1, y_2(b) = B_2,$$

where C, A_1, A_2, B_1 and B_2 are constants.

b. Use the results of part a. to maximise $(x(t), y(t) \in C^2[t_0, t_1])$

$$J(x,y) = \frac{1}{2}\int_{t_0}^{t_1}(x\dot{y} - y\dot{x})dt$$

subject to

$$\int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = l.$$

Note: Here J represents the area enclosed by a curve with parametric equations $x = x(t), y = y(t)$, with the constraint fixing the length of the curve. It is sufficient here to demonstrate that the curves are circles.

5. Consider the displacement field for torsion of a cylinder of the form

$$u_1 = -\alpha X_2 X_3, u_2 = \alpha X_1 X_3, u_3 = \alpha w(X_1, X_2).$$

Here we assume that the axis of the cylinder is in the 3-direction for $0 \leq X_3 \leq L$.

a. Show that the equilibrium equations reduce to

$$\frac{\partial^2 w}{\partial X_1^2} + \frac{\partial^2 w}{\partial X_2^2} = 0.$$

b. If the lateral surface is assumed to be stress-free show that the boundary conditions (using a standard notation) reduce to

$$\frac{\partial w}{\partial n} = \frac{1}{2}\frac{d}{ds}(X_1^2 + X_2^2).$$

c. Considering the moment on the end-face, show that the warping reduces the stiffness, in the case of a non-circular surface, from that of the circular case.

6. If the deformation of a cylindrical body is such that there is no axial component of the displacement and nothing depends on the axial coordinate, then the body is said to be in a state of *plane strain*. We may thus introduce the function $\phi(X_1,X_2)$ such that

$$T_{11} = \frac{\partial^2\phi}{\partial X_2^2}, T_{12} = -\frac{\partial^2\phi}{\partial X_1\partial X_2}, T_{22} = \frac{\partial^2\phi}{\partial X_1^2},$$
$$T_{33} = \nu\left(\frac{\partial^2\phi}{\partial X_1^2} + \frac{\partial^2\phi}{\partial X_2^2}\right).$$

In this case the compatibility equations reduce to

$$\frac{\partial^4\phi}{\partial X_1^4} + 2\frac{\partial^4\phi}{\partial X_1^2\partial X_2^2} + \frac{\partial^4\phi}{\partial X_2^4} = 0.$$

Consider the stress function $\phi(X_1,X_2) = \frac{1}{6}X_2^3$.

a. Obtain the stresses for the state of plane strain;

b. if the stresses of those of part a. are those inside a rectangular prism bounded by

$$X_1 = 0, X_1 = \ell, X_2 = \pm h/2 \text{ and } X_3 = \pm b/2$$

find the surface tractions on the boundaries;

c. if the boundary surfaces $X_3 = \pm b/2$ are traction-free, find the solution.

7. If **u** is the displacement vector in a linear elastic solid, the equations of motion (in the absence of body forces) may be written in the form

$$(\lambda+\mu)\nabla(\nabla.\mathbf{u}) + \mu\nabla^2\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

where λ, μ are the Lamé constants and ρ is the mass density.

a. If **u** is expressed in the form $\mathbf{u} = \nabla\phi + \nabla\times\Psi, \nabla.\Psi = 0$ where ϕ, Ψ are scalar and vector potentials, show that the equations of motion are satisfied provided that both satisfy the wave equation

$$c^2\nabla^2\chi = \frac{\partial^2\chi}{\partial t^2}.$$

Find the wave speed in each case.

b. Consider the displacement

$$u_1 = u_2 = 0, u_3 = A\cos(pX_2)\cos\left(\frac{2\pi}{\ell}(X_1 - ct)\right)$$

i. Show that this displacement is an equivoluminal motion.

ii. From the equations of motion determine the phase velocity in terms of p, ℓ, ρ and μ.

iii. This displacement is used to describe a type of waveguide that is bounded by the planes $X_2 = \pm h$. Find the phase velocity if these planes are traction-free.

Note: You may use the identity $\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla.\mathbf{a}) - \nabla^2\mathbf{a}$