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Ollscoil na hÉireann, Galway
National University of Ireland, Galway

Semester II Examinations, 2004/2005

Exam Code(s)	3CS1,3CS2,1EM1
Exam(s)	Third Science
Module Code(s)	MP364
Module(s)	Methods of Mathematical Physics II
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Micheál Ó Confhaola
	Dr. M.G. Meere
Instructions:	Attempt **THREE** questions
Duration	**TWO HOURS**
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	2 (excluding this page)
Department(s)	Mathematical Physics

1. Consider the following initial boundary value problem for a function $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L, t > 0,$$

$$u = 0 \text{ on } x = 0, t > 0,$$

$$\frac{\partial u}{\partial x} + \gamma u = 0 \text{ on } x = L, t > 0,$$

$$u = f(x) \text{ at } t = 0, 0 < x < L,$$

where $f(x)$ is a given smooth function and α^2, L, γ are positive constants.

a. Letting $u(x,t) = X(x)T(t)$, show that

$$X'' + \lambda X = 0,\ X(0) = 0,\ X'(L) + \gamma X(L) = 0,$$

and $T' + \alpha^2 \lambda T = 0$ where λ is the separation constant.

b. For $\lambda > 0$, show that the eigenvalues λ are determined as the solution to

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\, L) + \gamma \sin(\sqrt{\lambda}\, L) = 0.$$

Suggest graphically that this equation has an infinity of solutions for λ.

c. Construct the eigenfunctions of the above problem and hence obtain the solution for u.

2. Consider the following two point boundary value problem for a function $y(x)$:

$$L(y) = \lambda r(x) y,\ 0 < x < 1,$$

$$a_1 y(0) + a_2 y'(0) = 0,\ b_1 y(1) + b_2 y'(1) = 0$$

where

$$L(y) = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y$$

and where $p(x), q(x), r(x), a_1, a_2, b_1, b_2$ satisfy all of the usual conditions for the above problem to be regular. You may assume that $(L(u), v) = (u, L(v))$ for any two sufficiently smooth functions $u(x)$ and $v(x)$ satisfying the boundary conditions of this problem; here $(\cdot,\cdot)$ denotes the usual inner product.

a. Show that the eigenvalues of the above problem are real.

b. Suppose that $\phi_1(x)$ and $\phi_2(x)$ are eigenfunctions of the above problem with corresponding distinct eigenvalues λ_1 and λ_2, respectively. Show that (orthogonality)

$$\int_0^1 r(x)\phi_1(x)\overline{\phi_2(x)}dx = 0.$$

c. Determine the normalised eigenfunctions of the following problem:

$$y'' + \lambda y = 0,\ y'(0) = y'(1) = 0.$$

Determine the expansion of $f(x) = x$ in terms of these eigenfunctions.

3. Consider the following boundary value problem for Laplace's equation in a rectangular region:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } 0 < x < a,\ 0 < y < b,$$

$$u = 0 \text{ on } x = 0 \text{ for } 0 < y < b,$$

$$u = 0 \text{ on } y = 0 \text{ and } y = b \text{ for } 0 < x < a,$$

$$u = f(y) \text{ on } x = a \text{ for } 0 < y < b,$$

where $u = u(x,y)$, a,b are positive constants and $f(y)$ is a given smooth function. Solve this problem using the method of separation of variables.

4. Consider the following initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0,$$

$$u = f(x) \text{ at } t = 0, -\infty < x < \infty$$

where $u = u(x,t)$, K is a positive constant and $f(x)$ is a given smooth function. The intervals $-\infty < x < \infty$ and $t > 0$ are split into the uniform grids

$$\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots \text{ and } 0 = t_0 < t_1 < t_2 < \ldots$$

where $x_j = j(\Delta x), j = \ldots, -2, -1, 0, 1, 2, \ldots$ and $t_k = k(\Delta t), k = 0, 1, 2, \ldots$ with a view to solving the problem numerically using finite difference techniques. The numerical approximation to u at gridpoint (x_j, t_k) is denoted by U_j^k.

a. Using standard finite difference approximations for the derivatives, derive the following explicit time-stepping scheme for solving the above problem:

$$U_j^{k+1} = (1 - 2Kh)U_j^k + Kh(U_{j-1}^k + U_{j+1}^k)$$

where $h = \Delta t/(\Delta x)^2$.

b. The error in the implementation of the above difference scheme is denoted by $\bar{e}_j^k$. These errors also satisfy the scheme. By considering a representative Fourier term $\bar{e}_j^k = \exp(i(\theta j + \lambda k))$ $(i^2 = -1)$ where θ and λ are constants, show that

$$\exp(i\lambda) = 1 - 4Kh\sin^2(\theta/2),$$

and hence show that the explicit scheme is von Neumann stable only for $Kh \leq 1/2$.

c. The following scheme is obtained by using centred difference approximations for the time derivative in the heat equation:

$$U_j^{k+1} = U_j^{k-1} + 2Kh(U_{j+1}^k - 2U_j^k + U_{j-1}^k).$$

Show that this scheme is unstable for all $h > 0$.

5. a. Let $f(x)$ and $g(x)$ be real valued functions defined on $-\infty < x < \infty$ having Fourier transforms $\overline{f(c)}$ and $\overline{g(c)}$, respectively. Show that (the convolution theorem):

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{f(c)}\,\overline{g(c)}e^{-icx}dc = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)g(x-u)du.$$

b. Using the convolution theorem displayed in part **a.** of this question for the functions

$$f(x) = \begin{cases} 1 \text{ for } |x| \leq 1, \\ 0 \text{ for } |x| > 1, \end{cases}$$

and $g(x) = \exp(-|x|)$, show that

$$\int_{-\infty}^{+\infty}\frac{\sin(c)e^{-icx}}{c(1+c^2)}dc = \pi(1 - e^{-1}\cosh(x)).$$

Note: The Fourier transform and Fourier inverse transform of a function $f(x)$, $-\infty < x < \infty$, are

$$\overline{f(c)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{icx}f(x)dx \text{ and } f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-icx}\overline{f(c)}dc.$$