Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

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Summer Examinations, 2004/2005

Exam Code(s)	4BE1
Exam(s)	Fourth Civil Engineering
Module Code(s)	MP452
Module(s)	Applied Mathematics
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Micheál Ó Confhaola
	Dr. Pat O' Leary
	Dr. M.G. Meere
Instructions:	Attempt **FIVE** questions
Duration	**THREE HOURS**
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	3 (excluding this page)
Department(s)	Mathematical Physics

1. a. Determine the $C^2[0,1]$ curve with fixed end points $(0,0)$, $(1,1)$ which gives

$$J = \int_0^1 (y')^2/x^3 dx$$

a minimum value, assuming a minimum exists. What is the minimum value?

b. Find the extremal for the following functional satisfying the given boundary conditions:

$$J(y) = \int_0^1 (1 + (y'')^2) dx \text{ with } y \in C^4[0,1] \text{ and } y(0) = 0, y'(0) = 1, y(1) = 1, y'(1) = 1.$$

2. a. Consider the functional

$$J(y) = \int_0^{x_2} y^{1/2}(1 + (y')^2)^{1/2} dx$$

with $y \in C^2[0,x_2]$ such that $y(0) = y_1, y(x_2) = y_2$, x_2, y_1, y_2 being fixed positive constants. Prove that the curves of the foregoing type giving the functional a stationary value are of the parabolic type

$$2(cy - 1)^{1/2} = cx + d$$

where c, d are suitable constants.

Hint: The Euler-Lagrange equation for a functional of the form

$$J(y) = \int_0^{x_2} F(y,y') dx$$

has first integral

$$F - y' \frac{\partial F}{\partial y'} = \text{constant.}$$

b. Find the extremal for

$$J(y) = \int_0^1 (y^2 + (y')^2) dx,$$

subject to

$$y(0) = 1 \text{ and } y(1) \text{ unspecified.}$$

3. a. Derive a necessary condition for an extremum for the following isoperimetric problem. Minimize ($y \in C^2[a,b]$)

$$J(y) = \int_a^b F(x,y,y') dx$$

subject to

$$\int_a^b G(x,y,y') dx = C$$

and

$$y(a) = A, y(b) = B$$

where A, B and C are constants.

b. Find extremals for the isoperimetric problem ($y \in C^2[0,\pi]$)

$$J(y) = \int_0^\pi (y')^2 dx, \; y(0) = y(\pi) = 0,$$

subject to

$$\int_0^\pi y^2 dx = 1.$$

4. **a.** Let $y = y(x)$, $y \in C^2[0,l]$ be the unique solution of the two point boundary value problem

$$\frac{d}{dx}\left(T(x)\frac{dy}{dx}\right) - k(x)y = -w(x),\ 0 < x < l,$$
$$y(0) = y(l) = 0,$$

where $T(x) \in C^1[0,l]$ and $T(x) > 0$ in $[0,l]$, and where $k(x), w(x) \in C[0,l]$ and $k(x) > 0$, $w(x) \geq 0$ in $[0,l]$. Show that

$$\int_0^l \left\{\frac{(U' + w)^2}{k} + \frac{U^2}{T}\right\}dx \geq \int_0^l wy\,dx,$$

where U is any $C^1[0,l]$ function.

b. Consider the boundary value problem:

$$y'' - y = -1,\ 0 < x < \pi/2,$$
$$y(0) = y'(\pi/2) = 0.$$

Noting the bounds

$$\int_0^{\pi/2}\{(U' + 1)^2 + U^2\}dx \geq \int_0^{\pi/2} y\,dx \geq \int_0^{\pi/2}\{2Y - Y'^2 - Y^2\}dx$$

where $Y(0) = 0$, $U(\pi/2) = 0$ and U, Y are sufficiently smooth, calculate the 'best' Y of the type $Y = \beta \sin x$. Calculate the 'best' U of the type $U = \gamma \cos x$. Compute the corresponding bounds for $\int_0^{\pi/2} y\,dx$.

5. Consider the displacement field for torsion of a cylinder of the form

$$u_1 = -\alpha X_2 X_3,\ u_2 = \alpha X_1 X_3,\ u_3 = \alpha w(X_1, X_2).$$

Here we assume that the axis of the cylinder is in the 3-direction for $0 \leq X_3 \leq L$.

a. Show that the equilibrium equations reduce to

$$\frac{\partial^2 w}{\partial X_1^2} + \frac{\partial^2 w}{\partial X_2^2} = 0.$$

b. If the lateral surface is assumed to be stress-free show that the boundary conditions (using a standard notation) reduce to

$$\frac{\partial w}{\partial n} = \frac{1}{2}\frac{d}{ds}(X_1^2 + X_2^2).$$

c. Considering the moment on the end-face, show that the warping reduces the stiffness, in the case of a non-circular surface, from that of the circular case.

6. If the deformation of a cylindrical body is such that there is no axial component of the displacement and nothing depends on the axial coordinate, then the body is said to be in a state of *plane strain*. We may thus introduce the function $\phi(X_1,X_2)$ such that

$$T_{11} = \frac{\partial^2\phi}{\partial X_2^2}, T_{12} = -\frac{\partial^2\phi}{\partial X_1 \partial X_2}, T_{22} = \frac{\partial^2\phi}{\partial X_1^2},$$
$$T_{33} = \nu\left(\frac{\partial^2\phi}{\partial X_1^2} + \frac{\partial^2\phi}{\partial X_2^2}\right).$$

In this case the compatibility equations reduce to

$$\frac{\partial^4\phi}{\partial X_1^4} + 2\frac{\partial^4\phi}{\partial X_1^2\partial X_2^2} + \frac{\partial^4\phi}{\partial X_2^4} = 0.$$

Consider the stress function $\phi(X_1,X_2) = \frac{1}{6}X_2^3$.

a. Obtain the stresses for the state of plane strain;

b. if the stresses of those of part **a.** are those inside a rectangular prism bounded by

$$X_1 = 0, X_1 = \ell, X_2 = \pm h/2 \text{ and } X_3 = \pm b/2$$

find the surface tractions on the boundaries;

c. if the boundary surfaces $X_3 = \pm b/2$ are traction-free, find the solution.

7. If $\mathbf{u}$ is the displacement vector in a linear elastic solid, the equations of motion (in the absence of body forces) may be written in the form

$$(\lambda + \mu)\nabla(\nabla.\mathbf{u}) + \mu\nabla^2\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

where λ, μ are the Lamé constants and ρ is the mass density.

a. If $\mathbf{u}$ is expressed in the form $\mathbf{u} = \nabla\phi + \nabla\times\Psi, \nabla.\Psi = 0$ where ϕ, Ψ are scalar and vector potentials, show that the equations of motion are satisfied provided that both satisfy the wave equation

$$c^2\nabla^2\chi = \frac{\partial^2\chi}{\partial t^2}.$$

Find the wave speed in each case.

b. Consider the displacement

$$u_1 = u_2 = 0, u_3 = A\cos(pX_2)\cos\left(\frac{2\pi}{\ell}(X_1 - ct)\right)$$

i. Show that this displacement is an equivoluminal motion.

ii. From the equations of motion determine the phase velocity in terms of p, ℓ, ρ and μ.

iii. This displacement is used to describe a type of waveguide that is bounded by the planes $X_2 = \pm h$. Find the phase velocity if these planes are traction-free.

Note: You may use the identity $\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla.\mathbf{a}) - \nabla^2\mathbf{a}$