G 1069

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway
Semester Two Examinations, 2004/2005

Exam Code(s)	3BS5, 4BS4, 4FM2
Exam(s)	Third and Fourth Science
Module Code(s)	MP491
Module(s)	Non-Linear Systems
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan
Internal Examiner(s)	Dr. M. Ó Confhaola
	Professor T.N. Sherry

Instructions: Answer THREE questions.
All questions carry the same marks.

Duration	2 hours
No. of Answer books	1

Requirements

Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	

No. of Pages	3
Department(s)	Mathematical Physics

1. Answer each of the following three parts:

(a) The one-parameter family of non-linear ordinary differential equations

$$\frac{dx}{dt} = 1 - ax^2$$

where a is a real parameter, is characterised by a bifurcation. Investigate the nature of the bifurcation, sketch the corresponding bifurcation diagram and indicate on the diagram the behaviour of the non-equilibrium solutions.

(b) Classify the equilibrium point, at $(0,0)$, of the following two-dimensional system of linear ordinary differential equations

$$\frac{dx}{dt} = 4x + 2y\,, \quad \frac{dy}{dt} = -2x + y \ .$$

Using *either* the method of isoclines *or* the Jordan normal form approach, sketch the xy-phase portrait for this problem.

(c) The two-dimensional system of non-linear ordinary differential equations

$$\frac{dx}{dt} = ax - y - x\left(2x^2 + 5y^2\right) \ , \quad \frac{dy}{dt} = x + ay - y\left(2x^2 + 5y^2\right)$$

possesses a limit cycle solution for certain values of the parameter a. Investigate the nature of the Hopf bifurcation that occurs at the critical value of a and identify what that critical value is.

2. Consider the two-dimensional system of non-linear ordinary differential equations

$$\frac{dx}{dt} = x + y\,, \quad \frac{dy}{dt} = -\ x +\ y + xy\,.$$

(a) Locate and classify all the equilibrium points in the associated phase plane using linearisation methods.

(b) Discuss briefly what is meant by the direction field for this problem. Explain briefly how isoclines are useful?

(c) Identify, and sketch separately, the isoclines corresponding to $k = 0, 1, -1$ and ∞ for this non-linear problem.

(d) By using all four isoclines together, sketch a sufficient number of phase plane curves to illustrate the phase plane portrait for this problem.

3. Answer each of the following three parts:

(a) Explain what is meant by (i) the index of an equilibrium point and (ii) a limit cycle for a two-dimensional system of non-linear ordinary differential equations.

(b) Explain how the Poincare index theorem and the Poincaré-Bendixson theorem can be used to establish the existance of a limit cycle.

(c) By considering the solution path directions across the topographic system of concentric circles

$$x^2 + y^2 = \text{constant}$$

show that there is a limit cycle for the system

$$\frac{dx}{dt} = x - y - x\left(x^2 + 3y^2\right) \ , \quad \frac{dy}{dt} = x + y - y\left(3x^2 + y^2\right)$$

and that this limit cycle is constrained within the annulus

$$\frac{1}{\sqrt{2}} \leq \sqrt{x^2 + y^2} \leq 1 \ .$$

You may assume that $(0,0)$ is the only equilibrium point of the system of equations.

4. The one-parameter difference equation $x_{n+1} = a + x_n^2$ is obtained from the one-parameter map

$$f\ (x) = a + x^2 \ .$$

Investigate for what values of a the equation has fixed points and determine the stability properties of the fixed points you identify.

Investigate, also, for what values of a the equation has period-2 points, and what those points are. For what values of a are the resulting period-2 orbits stable?

Sketch, reasonably carefully, as much of the bifurcation diagram for the equation as you can. What do you think happens as a passes through the critical value at which the period-2 orbits become unstable?

Explain what you expect to occur eventually in iterations of this mapping in each of the following cases: *(a)* $a = -\frac{1}{2}$, $x_0 = 1.0$, and (b) $a = -\frac{9}{8}$, $x_0 = 1.0$.

5. The one-parameter family of three dimensional non-linear ordinary differential equations

$$\frac{dx}{dt} = 6y - 6x \ , \quad \frac{dy}{dt} = rx - y - xz \ , \quad \frac{dz}{dt} = xy - 2z$$

undergoes a sequence of bifurcations as the parameter r (> 0) increases in value.

(a) Show that there is only one equilibrium point for $r < 1$ but that there are three equilibrium points for $r > 1$.

(b) Show, further, that the $r < 1$ equilibrium point is asymptotically stable for $r < 1$ and unstable for $r > 1$.

(c) Show that the eigenvalue equation for the Jacobian matrix at the additional equilibrium points is

$$\lambda^3 + 9\lambda^2 + 2\,(r+6)\,\lambda + 24\,(r-1) = 0$$

and, hence, deduce that the two additional equilibrium points are asymptotically stable only for $1 \lesssim r < 22$.

(d) What special feature occurs in the solutions to the above system of equations corresponding to values of r just greater than the critical value $r_c = 22$?