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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER I EXAMINATIONS 2005 – 2006

CS 452 – Stochastic Models for Communication I

EXTERNAL EXAMINER: Prof. T.C. Bailey
INTERNAL EXAMINER: Dr. J. N. Sheahan

INSTRUCTIONS:

Time Allowed: **Two** hours.

Answer THREE questions. Each question carries 34 marks.

AIDS ALLOWED: Only the following text is allowed in the Examination: "*Stochastic Modeling in Broadband Communications Systems*" by Ingemar Kaj.

1.

(a) *[8 marks]*
In a communication system with discrete (slotted) time, in which cells of uniform size arrive at a buffered node for transmission to a link, let X_n be the number of cells arriving at the node during slot number n, and assume that the X_n, $n \geq 1$, are iid. Also let N_n be the number of cells in the system (in buffer or being transmitted) and assume that X_{n+1} is independent of $N_1, \ldots, N_n$. Is $\{X_n, n \geq 1\}$ a Markov chain? What about $\{N_n, n \geq 1\}$? Justify your answers.

(b) *[8 marks]*
A certain communication system is modelled by a Markov chain $\{X_n, n \geq 1\}$ with three states and transition probability matrix given by

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

Accept that the stationary distribution of this chain is $\underline{\pi}^T = (\pi_1, \pi_2, \pi_3)^T$ is $(0.3, 0.4, 0.3)^T$. Examine each of statements (i)–(iv) separately and establish if it is true or false.

(i) the chain is aperiodic
(ii) the chain is irreducible
(iii) $\underline{\pi}^T = \underline{\pi}^T\mathbf{P}$
(iv) $\lim_{n\to\infty} P^n_{ij} = \pi_j$, $j = 1, 2, 3$, and convergence take place independently of the initial state.

Question 1 is continued on the next page.

(c) *[8 marks]*

Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables with $P(X_n = j) = a_j, j = 0, \pm 1, \pm 2, \ldots$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. Show that $\{S_n, n \geq 1\}$ is a Markov chain and derive its transition probabilities.

~~**(d)** *[5 marks]*~~

Suppose that messages arrive at a transmission link according to a Poisson process of intensity $\lambda = \frac{1}{2}$. Suppose that the lengths $L_i, i = 1.2, \ldots$ of messages are iid with an exponential distribution with mean 2. The link has capacity $c = 4$ bits per second. Let N_∞ be the number of messages in the system when the system is in steady state. What are $E(N_\infty)$ and $Var(N_\infty)$? [See Kaj pages 35 and xiv.]

(e) *[5 marks]*

An M/M/1/K single-server loss system is equipped with a service node capable of handling 12,000 service requests per hour, and it has been estimated that under typical circumstances the traffic intensity demand on the system is given by $\rho = 1/2$. If we require that at most 200 arriving requests are lost per hour, what is the minimum acceptable buffer size?

2.

(a) *[17 marks]*

A binary source sends a sequence $X_1, X_2, \ldots$ of zeros and ones, where the X_i are iid Bernoulli with $p = P(X_i = 1)$, $0 < p < 1$. Because of transmission disturbances and digital-to-analog conversion, the sequence received is $Y_1, Y_2, \ldots$ where $Y_i = X_i + U_i$ and the U_i are iid $N(0, \sigma^2)$ with $\sigma^2 = \frac{1}{3}$. The disturbances are assumed independent of the X_i. If $Y_i > \frac{1}{2}$, the receiver interprets Y_i as a binary digit $X_i = 1$, and as a 0 otherwise (i.e. the receiver assumes the ith signal sent is a 1 if the received signal exceeds $\frac{1}{2}$, and as a 0 otherwise).

(i) Find $P(Y_i > \frac{1}{2})$ as a function of p .

(ii) Estimate p if 162 of 424 observations at the receiver are digit 1 and the rest are digit 0, when $\sigma^2 = \frac{1}{3}$.

Note: If Z has the $N(0,1)$ distribution, then $P\left(Z < \frac{\sqrt{3}}{2}\right) = 0.866025$.

(iii) For each i, find $E(Y_i | X_i = 1)$, $E(Y_i | X_i = 0)$, and $E(Y_i | X_i)$.

(iv) Obtain $E(Y_i)$ and $Var(Y_i)$ by two ways.

(v) If $S_n = \sum_{i=1}^{n} Y_i$, find the mean and variance of S_n/n.

(b) *[17 marks]*

Let the counting process $N := \{N_k : k = 0, 1, 2, \ldots\}$ be defined by $N_k = \sum_{i=1}^{k} X_i$ where the X_i are iid Bernoulli random variables with $p = P(X_i = 1)$, $0 < p < 1$. Further let K be a random variable with the positive geometric distribution

$P(K = k) = f_K(k) = \theta(1 - \theta)^{k-1}, k = 1, 2, \ldots$ [see Kaj page (xiv)]

and assume that K and N are independent.

(i) Is $N = \{N_k : k = 1, 2, \ldots\}$ a Markov chain?

(ii) Find $E(N_K)$ and $Var(N_K)$.

(iii) Find the probability generating function $E(z^{N_K}), 0 < z < 1$.

(iv) Find the distribution of N_K.

(v) Suggest an application of the random variable N_K.

3.

(a) *[17 marks]*

Let $\{X_n : -\infty < n < \infty\}$ be an infinite sequence of Bernoulli random variables with $P(X_n = 1) = p = P(\text{a computer job begins at time } n)$ and $P(X_n = 0) = 1 - p = P(\text{a computer job does not begin at time } n)$. Further, let T_i be the length of job i and let A_n be the number of jobs in the system at time n, $n = 1, 2, \ldots$.

(i) If T_i are iid each with a geometric distribution $P(T_i = k) = f_{T_i}(k) = (1-\alpha)\alpha^{k-1}$, $k = 0, 1, 2, \ldots$ [see Kaj page (xiv)], find $E(A_n)$.

Hint: Show first that $A_n = \sum_{i=-\infty}^{n} X_i 1_{\{T_i > n-i\}}$ where $1_{\{T_i > n-i\}} = \begin{cases} 1, \text{ if } T_i > n-i \\ 0, \text{ otherwise.} \end{cases}$

(ii) If instead all $T_i = c$ (constant), then find the probability distribution of A_n.

(b) *[17 marks]*

Recall that a renewal process is a sequence $\{U_n, n = 1, 2, \ldots\}$ of nonnegative iid random variables with a common distribution function F. Assume that $F(0) = P(U_n \le 0) = P(U_n = 0)$ satisfies $F(0) < 1$.
Think of U_n as the time between the $(n-1)$st and nth event (renewal). Let $v = E(U_n) = \int_0^\infty x dF(x)$ be the mean time. Define $T_0 = 0$ and $T_n = \sum_{i=1}^{n} U_i$, $n \ge 1$, so that T_n is the time of the nth event.

(i) Let N_t be the number of renewals by time t. Show that $N_t = \sup\{n : T_n \le t\}$.
(ii) Prove that an infinite number of renewals cannot occur in a finite time.
(iii) Derive the distribution $P(N_t = n)$, $n = 0, 1, 2, \ldots$ of N_t in terms of the distribution function $F_n(t)$, of $\sum_{i=1}^{n} U_i$.
Hint: Accept that if X_1 and X_2 are iid with distribution function F, then the distribution function $P(X_1 + X_2 \le a), -\infty < a < \infty$, of their sum is $\int_{-\infty}^{\infty} F(a-y)dF(y)$, $-\infty < a < \infty$. That is, the distribution function is $F * F$, the convolution of F with itself. Accept also that this extends to n iid random variables $X_1, X_2, \ldots, X_n$, i.e. the distribution function of $\sum_{i=1}^{n} X_i$ is

$$F_n = \underbrace{F * F * \ldots * F}_{n \text{ terms}} = F * F_{n-1} .$$

(iv) To assess the rate at which N_t goes to infinity, we examine $\lim_{t\to\infty} \frac{N_t}{t}$. Prove the following simple form of the Strong Law for Renewal Processes:
$P(\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{v}) = 1$, that is $\frac{N_t}{t} \xrightarrow{a.s.} \frac{1}{v}$ as $t \to \infty$.
Hint: Note that $T_{N_t} \le t < T_{N_t+1}$ where T_{N_t} represents the time of the last renewal *prior to or at* time t. You may accept that $N_\infty = \infty$ with probability 1, where $N_\infty = \lim_{t\to\infty} N_t$ is the total number of renewals that occur.

Question 3 is continued on the next page.

(v) [*Renewal Reward*] Suppose that packets arrive at a buffered node in accordance with a renewal process having a mean interarrival time v. Whenever there are N packets waiting in the buffer, they are all transmitted but at a cost to the Network of K Euro (to send them on a link to the internet). Suppose that the Network also incurs a cost of nc units per unit time whenever there are n packets waiting for transmission. If we say that a cycle is completed whenever N packets ~~are transmitted, then we have a renewal reward process and the expected length~~ of a cycle is the expected time required for N packets to arrive. Show that the average cost incurred per mean cycle length is $\frac{c(N-1)}{2} + \frac{K}{Nv}$.

4.

(a) *[19 marks]*

Consider the displacement X_t at time t (from its initial position) of a tiny particle immersed in a liquid. State clearly and in some detail (including definitions of any terms you use) the four assumptions that would make $\{X_t\}_{t\geq 0}$ a *Brownian motion* (Wiener) process.

(b) *[15 marks]*

[*Two-state chain*] Suppose that a binary source spends an exponentially distributed amount of time with intensity λ in state 0 before going to state (so the mean time in state 0 before going to state 1 is $1/\lambda$), and spends an exponentially distributed amount of time with intensity μ in state 1 before going to state 0. (In another context, think of a pay phone which is available – state 0 – or occupied – state 1). Assume that there is no queue, and that the system is in state 0 at time $t = 0$ (i.e. $P_{00}(0) = 1$).

(i) For each $t > 0$, find the probabilities $P_{00}(t) :=$ P(source in state 0 will be in state 0 at time t) and $P_{01}(t) :=$ P(source in state 0 will be in state 1 at time t).
Hint: Accept that $P'_{00}(t) = -\lambda P_{00}(t) + \mu P_{01}(t)$ and $P'_{11}(t) = \lambda P_{00}(t) - \mu P_{01}(t)$.

(ii) Find the probability that the source is in state 0 during the whole of the time interval $[0,t]$, $t > 0$.

(iii) Suppose that $\lambda > \mu$. After what time t is the probability that the source is in state 1 greater than the probability that it is in state 0?