GX1564

OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

EASTER EXAMINATIONS 1999

THIRD CIVIL, ELECTRONIC AND INDUSTRIAL ENGINEERING EXAMINATIONS

MM351 - MATHEMATICAL METHODS

Time allowed: *Three* hours.

Attempt *six* questions, *three* from each section.
Please use separate answer books for each section.

SECTION A

Professor J. Wiegold
Professor T. C Hurley
Dr. A. Christofides
Dr. J. Ward

1. Let G be a planar graph and let v, e and f be the number of vertices, edges and faces of G, respectively. Define the degree $d(y)$ of a face y of G and explain the formula

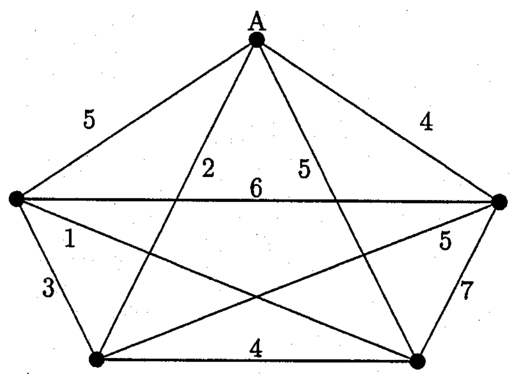
$$\sum_{y \in Y} d(y) = 2e,$$

where Y is the set of faces of G. Assuming Euler's formula

$$v - e + f = 2,$$

prove that $e \leq 3(v-2)$. Show that, if G is bipartite, then $e \leq 2(v-2)$.
Deduce that neither the complete graph K_5 nor the complete bipartite graph $K_{3,3}$ is planar.

2. Find the weight of a minimum spanning tree for the following weighted graph. Hence, or otherwise, find an upper and a lower bound for the travelling salesman problem in this graph, starting at A.

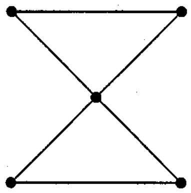


p.t.o.

3. Define the chromatic polynomial $P_G(k)$ of a graph and explain how it can be used to find the chromatic number $\chi(G)$ of G.

Show that the chromatic polynomial of a connected tree with n vertices is $k(k-1)^{n-1}$.

Explain how the chromatic polynomial of a graph can be determined by the Deletion-Contraction Process. Use this process to determine the chromatic polynomial of the following graph:



4. (a) Show that the image of a straight line or a circle under the linear fractional (Möbius) transformation

$$w = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0)$$

is a straight line or a circle.

(b) Find the Möbius transformation which sends

$$(i, 0, 1) \text{ to } \left(\frac{i}{2}, 0, \frac{1+i}{2}\right)$$

(c) Find the image of the half-strip $x < 0$, $0 < y < \pi$ under the transformation

$$w = e^z.$$

where $z = x + iy$.

5. Evaluate *two* of the following integrals

$$\text{(a)} \int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2} \qquad \text{(b)} \int_0^{\infty} \frac{dx}{x^4 + a^4} \qquad \text{(c)} \int_0^{\infty} \frac{\cos mx}{x^2+1} dx.$$

SECTION B

Prof. M.J. Sewell;
Prof. J.N. Flavin;
Dr. M.G. Meere.

6. (i) Define cartesian tensors (constant) of ranks one and two and write down the appropriate transformation laws in matrix form.
(ii) A tensor field of rank one has components $\mathbf{v} = (v_1(\mathbf{x}), v_2(\mathbf{x}), v_3(\mathbf{x}))$ in a cartesian frame $\mathbf{x} = (x_1, x_2, x_3)$. Show that its components in a frame $\mathbf{x}' = (x_1', x_2', x_3')$ where $\mathbf{x}' = \mathbf{R}\mathbf{x}$ and $\mathbf{R}$ is an orthogonal matrix, are given by
$$\mathbf{v}'(\mathbf{x}') = \mathbf{R}\mathbf{v}(\mathbf{R}'\mathbf{x}).$$
(iii) A tensor of rank two has components
$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
with respect to a given cartesian frame. The frame is rotated by an angle θ about its x_2 axis. Calculate the components of the tensor in the rotated frame.

7. Solve the following initial boundary value problem in $0 < x < \ell$, $t > 0$: $T(x,t)$ satisfies
$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2},$$
$$T(0,t) = 0, T(\ell,t) = 0,$$
$$T(x,0) = f(x)$$
where a is a given constant and $f(x)$ is a given function, by first proving that the separable variable solution of the partial differential equation
$$T(x,t) = X(x)\tau(t)$$
is such that
$$\tau(t) = \beta e^{-\lambda a^2 t}, X(x) = \gamma \cos \lambda^{1/2} x + \delta \sin \lambda^{1/2} x,$$
where $\lambda, \beta, \gamma, \delta$ are constants.
Deduce that, for $t > 0$,
$$|T(x,t)| \leq \frac{2}{\ell} \int_0^\ell |f(x)| dx \frac{e^{-\pi^2 a^2 t/\ell^2}}{1 - e^{-\pi^2 a^2 t/\ell^2}}.$$

8. (a) Prove that the separable variables solution of the two dimensional Laplace equation in plane polars (r, θ)
$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$
leads to ordinary differential equations of the type
$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0, \quad \frac{d^2\Theta}{d\theta^2} + \lambda\Theta = 0$$
where λ is a constant.
Prove that the equation for R can be transformed into one with constant coefficients by means of the transformation $r = e^t$. Hence determine its general solution for $\lambda \neq 0$. What is the most general form for Θ in these circumstances ?
(b) Find the steady state temperature T in the annular plate $b < r < a$, if the edges $r = a, r = b$ are held at constant temperatures T_a, T_b respectively. The quantities a, b are constants.

9. Consider the following initial value problem for the motion of an infinite string:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < +\infty, t > 0,$$

$$u = f_1(x)\,,\ \frac{\partial u}{\partial t} = f_2(x) \text{ at } t = 0, -\infty < x < +\infty.$$

Introducing the characteristic variables $\alpha = x - ct,\ \beta = x + ct$ and writing

$$u = f(\alpha) + g(\beta)$$

where f, g are arbitrary functions, show that

$$u = \frac{1}{2}(f_1(\alpha) + f_2(\beta)) + \frac{1}{2c}\int_{\alpha}^{\beta} f_2(s)ds.$$

Calculate the solution if $f_1(x) = 0, f_2(x) = \sin x$ and discuss the nature of the motion.

10. (a) Real valued functions $f(x), g(x)$ defined on $-\infty < x < +\infty$ are sufficiently smooth to admit fourier integral representations. Denoting by $\overline{f(c)}, \overline{g(c)}$ the fourier transforms of $f(x)$ and $g(x)$, respectively, show that (the convolution theorem)

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \overline{f(c)}\,\overline{g(c)}e^{-icx}dc = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u)g(x-u)du.$$

(b) Find the fourier transform of the function

$$f(x) = \begin{cases} e^{-x} \text{ if } x > 0, \\ 0 \text{ if } x < 0. \end{cases}$$

What does the inverse transform imply?

<u>Note.</u> The fourier transform of a function $f(x), -\infty < x < +\infty$ is defined by

$$\overline{f(c)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{icx}f(x)dx$$

with the inverse transform defined by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-icx}\overline{f(c)}dx.$$