GX1565

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

EASTER EXAMINATIONS 1999

B.E. DEGREEE EXAMINATION - MECHANICAL

MM353 - MATHEMATICAL METHODS

Time allowed: *Three* hours.

Attempt *six* questions, *three* from each section.
Please use separate answer books for each section.

SECTION A

Professor J. Wiegold
Professor T. C Hurley
Dr. A. Christofides

1. Let G be a planar graph and let v, e and f be the number of vertices, edges and faces of G, respectively. Define the degree $d(y)$ of a face y of G and explain the formula

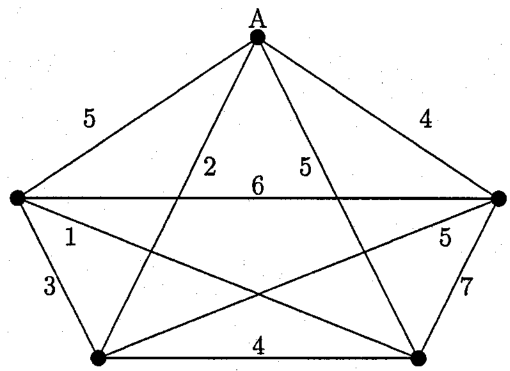
$$\sum_{y \in Y} d(y) = 2e,$$

where Y is the set of faces of G. Assuming Euler's formula

$$v - e + f = 2,$$

prove that $e \leq 3(v-2)$. Show that, if G is bipartite, then $e \leq 2(v-2)$.
Deduce that neither the complete graph K_5 nor the complete bipartite graph $K_{3,3}$ is planar.

2. Find the weight of a minimum spanning tree for the following weighted graph. Hence, or otherwise, find an upper and a lower bound for the travelling salesman problem in this graph, starting at A.

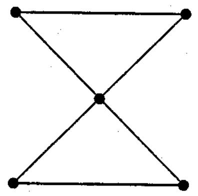


p.t.o.

3. Define the chromatic polynomial $P_G(k)$ of a graph and explain how it can be used to find the chromatic number $\chi(G)$ of G.

Show that the chromatic polynomial of a connected tree with n vertices is $k(k-1)^{n-1}$.

Explain how the chromatic polynomial of a graph can be determined by the Deletion-Contraction Process. Use this process to determine the chromatic polynomial of the following graph:



4. Let D be the network with vertices s, a, b, c, t and arcs and capacities given by the following table:

(s,a)	(s,b)	(a,b)	(a,c)	(a,t)	(b,c)	(c,t)
5	2	3	1	3	3	4

Apply the labelling algorithm to this network, starting with zero flow, and repeat the algorithm until you have obtained a maximum flow.

SECTION B

Prof. M.J. Sewell;
Prof. J.N. Flavin;
Dr. M.G. Meere.

5. Solve the following initial boundary value problem in $0 < x < \ell$, $t > 0$: $T(x,t)$ satisfies

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2},$$
$$T(0,t) = 0, T(\ell,t) = 0,$$
$$T(x,0) = f(x)$$

where α is a given constant and $f(x)$ is a given function, by first proving that the separable variable solution of the partial differential equation

$$T(x,t) = X(x)\tau(t)$$

is such that

$$\tau(t) = \beta e^{-\lambda\alpha^2 t}, X(x) = \gamma \cos \lambda^{1/2} x + \delta \sin \lambda^{1/2} x,$$

where $\lambda, \beta, \gamma, \delta$ are constants.
Deduce that, for $t > 0$,

$$|T(x,t)| \le \frac{2}{\ell} \int_0^\ell | f(x) | dx \frac{e^{-\pi^2\alpha^2 t/\ell^2}}{1 - e^{-\pi^2\alpha^2 t/\ell^2}}.$$

6. (a) Prove that the separable variables solution of the two dimensional Laplace equation in plane polars (r,θ)

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

leads to ordinary differential equations of the type

$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0, \quad \frac{d^2\Theta}{d\theta^2} + \lambda\Theta = 0$$

where λ is a constant.
Prove that the equation for R can be transformed into one with constant coefficients by means of the transformation $r = e^t$. Hence determine its general solution for $\lambda \neq 0$. What is the most general form for Θ in these circumstances ?
(b) Find the steady state temperature T in the annular plate $b < r < a$, if the edges $r = a, r = b$ are held at constant temperatures T_a, T_b respectively. The quantities a, b are constants.

7. Consider the following initial value problem for the motion of an infinite string:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < +\infty, t > 0,$$
$$u = f_1(x)\,, \frac{\partial u}{\partial t} = f_2(x) \text{ at } t = 0, -\infty < x < +\infty.$$

Introducing the characteristic variables $\alpha = x - ct$, $\beta = x + ct$ and writing

$$u = f(\alpha) + g(\beta)$$

where f, g are arbitrary functions, show that

$$u = \frac{1}{2}(f_1(\alpha) + f_2(\beta)) + \frac{1}{2c}\int_\alpha^\beta f_2(s)ds.$$

Calculate the solution if $f_1(x) = 0$, $f_2(x) = \sin x$ and discuss the nature of the motion.

8. (a) Real valued functions $f(x), g(x)$ defined on $-\infty < x < +\infty$ are sufficiently smooth to admit fourier integral representations. Denoting by $\overline{f(c)}, \overline{g(c)}$ the fourier transforms of $f(x)$ and $g(x)$, respectively, show that (the convolution theorem)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \overline{f(c)g(c)}\, e^{-icx}\, dc = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)g(x-u)\,du.$$

(b) Find the fourier transform of the function

$$f(x) = \begin{cases} e^{-x} \text{ if } x > 0, \\ 0 \text{ if } x < 0. \end{cases}$$

What does the inverse transform imply?

Note. The fourier transform of a function $f(x), -\infty < x < +\infty$ is defined by

$$\overline{f(c)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{icx} f(x)\,dx$$

with the inverse transform defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-icx}\, \overline{f(c)}\,dc.$$