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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER II EXAMINATIONS 1998-99

MA 484/6, 487 – STATISTICS

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Time Allowed: **Three** hours
Answer any *five* of the seven questions.
(All questions, but not necessarily parts therein, carry equal marks.)
This examination carries a weight of 82% of the course.

1.

Let $X_1, X_2, ..., X_n$ be i.i.d. $N(0, \theta)$ random variables. We desire to to use the likelihood ratio (LR) procedure to test the alternatives $H_0 : \theta \leq \theta_0$, $H_1 : \theta > \theta_0$ at size α.

a. **Show** that the maximum likelihood estimate of θ over the entire parameter space $\Omega = \{\theta \mid \theta > 0\}$ is $\frac{1}{n}\sum_{i=1}^{n} X_i^2$.

b. **Find** the (LR) test of the alternatives above. Ensure that you show how to make the test have size α. (Recall that the sum of squares of n independent standard normal variables is a χ^2_n variable.)

c. Suppose that in **b.** above we have $n = 5$, $\theta_0 = 1$ and $\alpha = 0.05$. If the data turns out to be $\underset{\sim}{x} = (-3, -2, 0, 2, 3)$, **what** is your decision? **What** is the power of your test if in fact $\theta = 13.3574$? *Note*: Two of the following chi-squared values are relevant:

$$\chi^2_{5,0.05} = 11.1,\ \chi^2_{5,\ 0.99} = 0.554,\ \chi^2_{5,\ 0.975} = 0.831,\ \chi^2_{5,0.95} = 1.145.$$

d. In **c.** above, **is** it necessary to have the actual 5 data points (or at least the sum of their squares) to perform the power calculation? Explain briefly.

2.

a. **State** *without* proof the Neyman-Pearson (NP) Lemma on testing a simple null hypothesis $H_0 : \underset{\sim}{\theta} = \underset{\sim}{\theta}_0$ against a simple alternative hypothesis $H_1 : \underset{\sim}{\theta} = \underset{\sim}{\theta}_1$.

b. **Show** that the NP test is unbiased.

c. Let $X_1, X_2, ..., X_n$ be i.i.d. $N(\theta, 1)$ random variables. **Find** explicitly the uniformly most powerful (UMP) test of size α of $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$, by first using the Neyman Pearson Lemma to test H_0 against a simple alternative in Ω_{H_1}.

d. **Establish** the fact that the family of joint densities (of the random sample vector) in **c.** above has the monotone likelihood ratio property, and, using this, along with a theorem of Karlin and Rubin (which you must state), **show** that the test in **c.** above is in fact the UMP level α test of the alternatives $H_0 : \theta \leq \theta_0$, $H_1 : \theta > \theta_0$.

3.

a. **Define** the *mean squared error* of an estimator $\hat{\theta}$ of a parameter θ, and **prove** that a minimum (uniformly in the unknown parameter θ) mean squared error estimator does not exist in general.

b. **Define** an unbiased estimator U of a parameter $\tau(\theta)$ and **give** separate examples, with verification, to show **any two** of (A), (B) and (C) below:

(A) there may be an infinite number of unbiased estimators in a given situation;

(B) there may be no unbiased estimator;

(C) an unbiased estimator may be absurd/ridiculous.

c. Let $X_1, X_2, ..., X_n$ be a random sample from an infinite population that has mean μ and (finite) variance σ^2. **Show** that $\bar{X}$ and $S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ are unbiased estimators of μ and σ^2, respectively.

d. **Show either that** any function of a complete statistic is complete **or else state and prove** a functional invariance result concerning maximum likelihood estimators.

Turn over ⇒

4.

a. Briefly **explain** the two concepts involved in the statement that a confidence interval $S(\underset{\sim}{x})$ is "*uniformly most accurate* at *confidence level* $1 - \alpha$."

b. Briefly **explain** how an uniformly most accurate confidence interval for a parameter θ can be obtained from a uniformly most powerful test about θ

c. If a 95% one-sided confidence intrerval for θ is $\theta < 75$, carefully **interpret** this interval. Do *not* write: "the probability is 0.95 that $\theta < 75$".

d. If a 95% confidence interval for the mean μ of a $N(\mu, \sigma_0^2)$ density turned out to be $65 < \mu < 69$, **would** the corresponding test of the alternatives $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$ lead to rejection of H_0 with $\alpha = 0.05$? (Assume that the same data set is used for the test as was used in deriving the confidence interval.) Justify your answer briefly.

e. Let $X_1, X_2, ..., X_n$ be a random sample from the density $f_X(x;\theta)$, $\theta \in \Omega$. If a sufficient statistic T exists and if a maximum likelihood estimator $\widehat{\theta}$ of θ also exists and is unique, **prove** that $\widehat{\theta}$ is a function of T.

5.

a. Let $X_1, X_2, ..., X_n$ be a random sample from a $U(0,\theta)$ density, $\theta > 0$.

i. **Find either** the likelihood ratio test of size α **or** the uniformly most powerful test of level α of $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$. (The two tests are the same.)

ii. **Find** the power function of the test in **i.** above.

b. Let $X_1, X_2, ..., X_n$ be a random sample from a population that has mean μ and (finite) variance σ^2. Use the *delta method* to **find** the approximate distribution of $\overline{X}^{1/2}$, the square root of the sample mean.

6.

a. **State** a result you know about when equality holds in the Cramér-Rao lower bound and use this result to **find** to the minimum variance unbiased estimator of the mean of a $N(\theta, \sigma_0^2)$ density. Also, **write down** the variance of this estimator.

b. Let $X_1, X_2, ..., X_n$ be a random sample from a Cauchy density $f_X(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]}, -\infty < x < \infty$ $(-\infty < \theta < \infty)$. **Find** the Cramér-Rao lower bound $[\tau'(\theta)]^2 / \left[-nE\frac{\delta^2}{\delta\theta^2}\ln f_X(x,\theta)\right]$ for the variance of any unbiased estimator of θ.

Hint: You may find it helpful to use the substitution $u = \tan\theta$ in the integral $\int_0^\infty (1 - u^2)/(1 + u^2)^3 du$, and accept that $\int_0^{\pi/2} \sin^2\theta d\theta = \pi/4$, $\int_0^{\pi/2} \sin^4\theta d\theta = 3\pi/16 = \int_0^{\pi/2} \cos^4\theta d\theta$.

7.

a. **State and prove** the Neyman Factorisation Theorem. (You may assume discrete random variables.)

b. **State** a condition that establishes if a statistic is *minimal* sufficient.

c. Let $X_1, X_2, ..., X_n$ be a random sample from a $N(\theta, 1)$ density.

i. **Show** that $T = \sum_{i=1}^n X_i$ is a complete and sufficient statistic

ii. **Find** the minimum variance unbiased estimator of θ^2.

J.N.S.