# OLLSCOIL NA hÉIREANN, GAILLIMH
# NATIONAL UNIVERSITY OF IRELAND, GALWAY

## SEMESTER II EXAMINATIONS 1998-99

## MA 387 – PROBABILITY

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Time Allowed: **Three** hours
**Answer question 1 and any *four* of questions 2–7.**
This examination carries a weight of 70% of the course.

1. **Insert** *numerical* answers in each of **a**. through **h**. below. Part marks will *not* be allotted in any part, so any work you may exhibit will *not* be examined. Full marks for any **six** parts answered correctly.

   **a.** If $f(x)$ is the density function of a continuous random variable $X$, **what** must $\int_{-\infty}^{\infty} f(x)dx$ equal?

   **b.** $X$ and $Y$ are independent and each has variance 1.0. **What** is $Var(X+Y)$ ?

   **c.** If $Var(X) = 4.0 = Var(Y)$ and $Var(X-3Y) = 37.0$ , **what** is $cov(X,Y)$ ?

   **d.** Incomes of people in a certain country have mean 10,000 and standard deviation 1,000 (– ignore units). Using Chebychev's inequality (– which is stated in **Q. 6. b.** below), at least **what** proportion of people earn between 8,000 and 12,000 ?

   **e.** Let $X$ have moment generating function $E(e^{tX}) = 1, -\infty < t < \infty$. Then it is easy to see that $X$ takes only one value (with probability 1), i.e. $X$ is actually a constant. **What** is this constant?

   **f.** A discrete random variable $X$ has cumulative distribution function

   $$F(x) = P(X \le x) = \begin{cases} 0, & x < 1 \\ 1/4, & 1 \le x < 2 \\ 3/4, & 2 \le x < 3 \\ 1, & x \ge 3. \end{cases} \quad \textbf{What is } P(1 < X \le 3.5)?$$

   **g.** $X$ and $Y$ are independent, each has mean 4 and each has variance 4. **What** is $E(X^2Y)$ ?

   **h.** The 4 people in an elevator will get off independently and each person has a probability of 1/2 of getting off at any one of the two floors above the ground floor. **What** is the expected number of stops made by the elevator above the ground floor?

2. The number of burnt cornflakes $X$ that appear on a conveyor belt in the time interval from 0 to $u$ has the Poisson distribution with density

   $$f_X(x) = P(X=x) = (\lambda u)^x e^{-\lambda u}/x!, \; x = 0,1,2,...,\infty.$$

   **i.** **Find** $E(X)$ and $Var(X)$ in terms of $\lambda$ and $u$. (Write $m$ for $\lambda u$ for short for the moment if you wish.)

   **ii.** **Find** the moment generating function of $X$.

   **iii.** **Derive** the (exponential) density of the time $U$ taken for the first burnt cornflake to arrive.

   **iv.** **Show** that $U$ satisfies $P(U > a+b \mid U > a) = P(U > b)$ for any positive constants $a$ and $b$.

   **v.** **Give** *without derivation* the formula for the density of a *discrete* random variable that has the memoryless property defined in **iv**. above.

   **vi.** **What** should $\lambda$ be in the density of $X$ above if it is desired that with probability 0.9, no burnt cornflake will appear in the time interval from $u = 0$ to $u = 1/2$ ?

3. **a.** Let $X$ have the uniform density on the interval $(0,1)$. Use the method of distribution functions *or* the method of transformations (– state which you intend to use ) to **derive** the density of $Y = \frac{-1}{\lambda}\ln(1-X)$ where $\lambda$ is a positive constant. **Identify** this density, and **give** briefly the practical importance of the question in computer random variate generation.

   **b.** Let $X \sim N(\mu, \sigma^2)$, so that the density of $X$ is $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\{\frac{-1}{2}(x-\mu)^2\}, -\infty < x < \infty.$

   **i.** **Show** that $f_X(x)$ is in fact a density function.

   **ii.** *Accept* that $Z := \frac{X-\mu}{\sigma}$ has the $N(0,1)$ density. **Show** that $U = Z^2$ has the $\chi_1^2$ density (the formula for which is, incidentally, a special case of the gamma density formula in **Q.4.** below.) *Note*: There should be *no* appearance of $\mu$ and $\sigma$ in your work!

   **iii.** The diameter $X$ of a random bolt from a certain machine has a normal distribution with mean 1 cm and standard deviation 0.02 cm. **Find** $P(X^2 - 2X + 1 > (0.02)^2)$ by (**iii a.**) using standard normal tables, noting that these tables give $P(Z > 1) = 0.1587$, and by (**iii b.**) using tables of the $\chi_1^2$ density, noting that these tables give $P(U > 1) = 0.3174$.

**Turn over** ⇒

4. Let $X$ have the Gamma $(\alpha, \beta)$ density, i.e., $f_X(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$

   i. **Show** that the moment generating function of $X$ is $M(t) = 1/(1 - \beta t)^\alpha, t < 1/\beta$.

   ii. Using $M(t)$, **find** $E(X)$ and $Var(X)$.

   iii. Let $X$ be *any* random variable with moment generating function $M(t)$, $0 < t < h$. Using the result of **Q.6. a.** below, **prove** that $P(X \geq a) \leq e^{-at}M(t)$ for all $t$ satisfying $0 < t < h$.

   iv. If $X$ has the Gamma$(\alpha, \beta)$ density, use the results of **i.** and **iii.** above to **find** the smallest number $c$ (depending only on $\alpha$) so that $P(X \geq 2\alpha\beta) \leq c$.

5. Tom and Mary will arrive *independently* at a nightclub sometime after midnight (time 0 hours). Tom's arrival time $X$ and Mary's arrival time $Y$ each have the exponential density with mean 1 hour. Thus

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}, \quad f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

   i. **Explain** briefly why the above information implies that the joint density of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & x > 0, y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

   ii. Whoever arrives first will wait for at most 1 hour for the arrival of the other. **Find** the probability that they meet by integrating $f_{X,Y}(x,y)$ over an appropriate region.

   iii. **Find** the density of $U = X - Y$.

   iv. Using your result in **iii.** above, **find** again the probability that Tom and Mary will meet.

   v. Using any *two* methods, **find** $E(X - Y)$.

6. **Answer any *four* of a. through f. below *and* answer *both* g. and h.**

   a. Let $u(X)$ be a nonnegative function of the (discrete or continuous – your choice) random variable $X$. If $E[u(X)]$ exists, **show** that for each constant $c > 0$, $P(u(X) \geq c) \leq E[u(X)]/c$.

   b. Using the result in **a.** above, **prove** Chebychev's inequality, i.e. if $X$ has finite variance $\sigma^2$, then prove that for any $k > 0$, $P(|X - E(X)| \geq k\sigma) \leq 1/k^2$.

   c. Recall that $X_1, X_2, ..., X_n$ is a random sample if the $X_i$ are *independent* and have *the same distribution.* **Explain** symbolically what each of these two means.

   d. Let $X_1, X_2, ..., X_n$ be a random sample from an infinite population that has mean $\mu$ and variance $\sigma^2$, and define $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. **Show** that $E(\bar{X}) = \mu$ and $Var(\bar{X}) = \sigma^2/n$.

   e. **State** and, using Chebychev's inequality, **prove** the Weak law of Large Numbers (WLLN).

   f. **State**, without proof, any version of the Central Limit Theorem (CLT) you know.

   g. **Which** of the WLLN and the CLT do you consider to be a "stronger result". Justify your answer briefly but clearly.

   h. If the CLT holds, **must** the WLLN be true? If so, prove. If false, give an example to show it is false.

7. Use the Central Limit Theorem in parts **ii.** and **iii.** below. *Note*: You will need two of the following tabulated numbers:

$$\text{If } Z \backsim N(0,1), \text{then } P(Z > 1) = 0.1587, P(Z > 2) = 0.0228, P(Z > 3) = 0.0013.$$

   i. Let $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli $(p)$ density. (Thus the $X_i$ are independent and each takes values 1 and 0 with probabilities $p$ and $1 - p$, respectively.) **Write down** (*without* derivation if you know them), the mean and variance of $\hat{p}$ = the sample proportion of ones (successes) = the sample mean = $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. (Of course your answers are a special case of the fomulae in **Q.6. d.** above.)

   ii. **Write down** the mean and variance of $\hat{p}$. **What** is the minimum number of Galwegians that should be sampled at random so that with probability (at least) 0.9544, $\hat{p}$ will not differ from $p$ by more than $\pm$ 0.04 ?

   iii. Airlines and hotels often grant reservations in excess of capacity to minimise losses due to no-shows. Suppose that the records of the Bates Motel show that, on average, only 80% (i.e. $p = 0.80$) of its guests will claim their reservation. If the Bates Motel accepts 64 room reservations for next Saturday night, and there are only 48 guest rooms in the motel, **what** is the approximate probability that all who arrive on Saturday night to claim a room will receive one?

J.N.S.