OLLSCOIL NA hÉIREANN, GAILLIMH
THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

FIRST SCIENCE EXAMINATION

MATHEMATICS [MA103]

MA103 — *ALGEBRA*

PASS
Second Paper

Professor J. Wiegold
Professor T. Hurley
Professor M.L. Newell

Time allowed: *three* hours.
Answer six questions.

1. Consider the matrices $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -4 \\ -2 & 5 \end{pmatrix}$.

a) Find A^{-1} and B^{-1}.

b) Show that $A^{-1}B^{-1}$ is not the inverse of AB.

c) Verify that $|AB| = |BA|$ ($|X|$ is the determinant of X).

2. Let A be the matrix $A = \begin{pmatrix} -1 & -5 \\ -2 & 2 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Write down matrices E and D with D diagonal, such that $AE = ED$, and hence calculate A^{10}.

c) For the transformation of the plane defined by A find

(i) The image of the line $x + y = 0$.
(ii) The line whose image is $x + 2y = 3$.

p.t.o.

3. a) Reduce the conic $3x^2 - 2y^2 + 6x + 4y + 3 = 0$ to standard form and sketch its graph.

b) Let $a \neq 1$ by an integer. Prove by induction that

$$\begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} a^n & \dfrac{a^n - 1}{a - 1} \\ 0 & 1 \end{pmatrix} \text{for all n} \geq 1.$$

3. a) Write $(1 - i)$ in polar form. Hence show that $(1 - i)^{16} = 256$.

b) Indicate on Argand diagrams points which satisfy:

(i) $\text{Re}(z) = -1$.

(ii) $\arg(z) = \dfrac{3\pi}{4}$.

(iii) $1 \leq |z - (1 + i)| < \sqrt{2}$.

c) Using De Moivre's Theorem, or otherwise, show that

$$\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$$

5. a) (i) Find the five distinct complex values of $1^{1/5}$.
(ii) Factorize $z^5 - 1$ into linear factors.

b) (i) Find $\sqrt{5 - 12i}$.
(iii) Solve $z^2 + (1 - 2i)z - 2(1 - i) = 0$.

6. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ -1 & -3 & 2 \end{pmatrix}$.

a) Find the matrix of cofactors of A.

b) Find $\det(A)$.

c) Use a) and b) to show that $A^{-1} = \begin{pmatrix} 17 & -7 & 2 \\ -7 & 3 & -1 \\ -2 & 1 & 0 \end{pmatrix}$.

d) Use the answer in part c) to solve:

$$\begin{aligned} x + 2y + z &= 1 \\ 2x + 4y + 3z &= 2 \\ -x - 3y + 2z &= -2 \end{aligned}$$

p.t.o.

7. In a certain region of Nigeria land is either classed as desert or scrub. Each year 20% of the scrub land turns to desert and 10% of the desert land becomes scrub. Let x_n be the number of hectares of desert in year n and y_n be the number of hectares of scrub, in year n.

a) Find the transition matrix, A, corresponding to this process.

b) Show that

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where x_0 and y_0 are the land areas in the initial year.

c) Show that 1 is an eigenvalue of A and find the other eigenvalue.

d) Find two linearly independent eigenvectors of A.

e) The total area of this region is 3,000,000 hectares. What area of land would be desert and what area would be scrub, in a steady state of this process?

f) Show that after a large number of years any initial distribution of land would tend to the steady state distribution.