GX1572

OLLSCOIL NA hÉIREANN, GAILLIMH
THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

FIRST ARTS EXAMINATION

MATHEMATICS [MA123]

MA123 — *ALGEBRA*

PASS
Second Paper

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Time allowed: *Three* hours.
Answer five questions.

1. Consider the matrix $A = \begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix}$.

a) Calculate A^2 and find the number k such that $A^2 - 8A = kI$, where I is the identity matrix.
b) Find the eigenvalues and eigenvectors of A. Hence write down a diagonal matrix D and a non-singular (invertible) matrix E such that $AE = ED$. Verify this equation and deduce that $A = EDE^{-1}$.

2. Consider the matrices $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

a) Calculate A^{-1} and use it to solve the matrix equation (i) and the system (ii):

(i) $XA = B$ (ii) $\begin{aligned} x + 3y &= 13 \\ 2x + y &= -4 \end{aligned}$

b) Write down the linear transformation defined by A. Find the image of the line $3x + 4y = 7$ under this transformation. Find also a point whose image is $(13, -4)$.

3. a) Prove by induction that $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$ for all $n \geq 1$.

b) Use the Euclidean algorithm to calculate $gcd\,(84,35)$ and express it in the form $84s + 35t$ with s and t in Z. Find a point with integer coordinates on the line $84x + 35y = 21$, and explain carefully why there is no such point on the line $84x + 35y = 41$.

p.t.o.

4. The sequences $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ satisfy the equations

$$x_{n+1} = \frac{4}{5}x_n + \frac{2}{5}y_n$$

$$\text{for } n \geq 0.$$

$$y_{n+1} = \frac{1}{5}x_n + \frac{3}{5}y_n$$

a) Write down the (transition) matrix A such that

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A\begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{and show that} \quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

b) Find the eigenvalues and eigenvectors of A and hence calculate A^n.

c) Suppose that $x_0 = \dfrac{1}{3}$ and $y_0 = \dfrac{2}{3}$. Calculate x_7 and say what happens to x_n as n gets large.

5. a) Explain what is meant by saying a Markov (Transition) matrix is regular. Show that if A is a Markov (Transition) matrix then so is A^2.

b) Identify and sketch the conic section

$$2x^2 + y^2 - 12x - 4y + 18 = 0.$$

6. a) Calculate A^{-1} where

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

b) Hence solve the system of linear equations

$$3x_1 + x_2 = 0$$

$$-2x_1 - 4x_2 + 3x_3 = 1$$

$$5x_1 + 4x_2 - 2x_3 = -1$$

7. (a) Find all the cube roots of -8 and hence factorize the polynomial $p(x) = x^3 + 8$ into real linear and quadratic factors.

(b) Verify the equation $(\sigma_j)^{-1} = \sigma_j$ for $j = 1, 2, 3$, for the following matrices (known as Pauli Spin matrices):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$