OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

FIRST UNIVERSITY EXAMINATION

MATHEMATICS [MA181]

HONOURS

First Paper

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Time allowed: *Three* hours.
Full marks for *six* questions.

1. (a) Find all the asymptotes and intercepts and sketch the graph of the function

$$\frac{8+2x-x^2}{x^2+4x+3}.$$

(b) Evaluate the following limits:

$$\text{(i)}\ \lim_{x\to 1}\frac{x^3-x^2+x-1}{x^4-1} \qquad \text{(ii)}\ \lim_{\theta\to 0}\frac{\sin^2\theta}{\sin 2\theta\,\sin 3\theta} \qquad \text{(iii)}\ \lim_{x\to\infty}\sqrt{x^4+6}-x^2.$$

2. (a) Find the derivative, y', in *two* of the following cases:

$$\text{(i)}\ y=\sqrt[4]{\frac{(x-2)^2(2x+3)}{(3x-1)^3}}; \qquad \text{(ii)}\ y=x^{\sin x}; \qquad \text{(iii)}\ x^3y^2-x\cos(x+y)=1.$$

(b) A wall 8 metres high is 1 metre from a house. Find the shortest ladder that will reach from the ground to the house when leaning over the wall.

p.t.o.

3. (a) Evaluate *two* of the following integrals:

(i) $\int \sin^5 x \cos^2 x \, dx$, (ii) $\int \frac{x\,dx}{x^2 - 3x - 4}$, (iii) $\int e^{-x} \sin(2x)\, dx$.

(b) Find a reduction formula for the integral

$$I_n = \int_0^{\pi/2} \cos^n x \, dx$$

and use it to evaluate I_6.

4. (a) Use the Limit Theorems carefully to show that

$$\lim_{x \to 5} (x^2 + 2x - 3) = 32 .$$

(b) Give the formal definition for $\lim_{x \to c} f(x) = \ell$. Use the definition to establish the limit in part (a).

(c) Give the formal definition for $\lim_{n \to \infty} a_n = \ell$ and show using the definition that

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 .$$

5. (a) State the Intermediate Value Theorem and use it to show that the equation $x = \cos x$ has a solution in $[0, \pi/2]$.

(b) The function f is continuous at c, and $f(c) > 0$. Prove that there exists $\delta > 0$ such that $f(x) > 0$ for all x in $(c - \delta, c + \delta)$.

(c) Outline a proof of the Intermediate Value Theorem.

6. (a) The function f is defined on $(1, 3)$ as follows:

$$f(x) = \begin{cases} ax^2 + x + c & \text{if } 1 < x \le 2 \\ 3x - 4 & \text{if } 2 < x < 3 . \end{cases}$$

Show that f is continuous on $(1, 3)$ if and only if $c = -4a$. What are the values of a and c if f is differentiable on $(1, 3)$?

(b) Prove the Mean Value Theorem, assuming Rolle's Theorem. The function f is such that $f'(x) = 0$ for all x in $(0, 1)$. Show that f is constant-valued on $(0, 1)$.

p.t.o.

7. (a) The function h is defined by $h(x) = (1+x^2)^{-1}$, and P is the partition of $[0,1]$ into five equal parts. Calculate the upper and lower Riemann sums $U(h,P)$ and $L(h,P)$, and verify that

$$L(h,P) \le \int_0^1 \frac{dx}{1+x^2} \le U(h,P).$$

(b) Let f be integrable on $[a,b]$ with $m \le f(x) \le M$ for all x in $[a,b]$. Explain why

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

(c) Suppose further that f is continuous on $[a,b]$. Show that

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable on (a,b), with $F'(x) = f(x)$.

8. (a) Show that the improper integral

$$\int_1^\infty \frac{dx}{x^3}$$

converges, and evaluate it.

(b) The sequence (a_n) is increasing and bounded above. Prove that it converges.

(c) Show that

$$\frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} \le \int_1^n \frac{dx}{x^3}.$$

Deduce that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.