OLLSCOIL NA hÉIREANN, GAILLIMH
THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

FIRST YEAR EXAMINATION

MATHEMATICS [MA180]

MA183 — *ALGEBRA*

HONOURS
Second Paper

Professor J. Wiegold
Professor T. Hurley
Dr. R.M. Dunwell
Dr. G. Pfeiffer

Time allowed: *Three* hours.
Answer six questions.

1. a) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 7 & -2 \\ -3 & 2 \end{pmatrix}$.

b) The eigenvalues of $A = \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix}$ are -1 and 5 and two eigenvectors corresponding to these values are $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. Use this information to:

(i) Write A in the form EDE^{-1} where D is a diagonal matrix.

(ii) Solve the recurrence relations

$$a_{n+1} = 4a_n + b_n$$
$$b_{n+1} = 5a_n$$

where $a_0 = 1$ and $b_0 = 2$.

c) Show that the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2;\ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - y \\ 2y \end{pmatrix}$ is linear.

p.t.o.

2. a) Let $A = \begin{pmatrix} 1 & -2 & 1 \\ 3 & -5 & 4 \\ 2 & -2 & 1 \end{pmatrix}$. Show that $A^{-1} = \frac{1}{3}\begin{pmatrix} -3 & 0 & 3 \\ -5 & 1 & 1 \\ -4 & 2 & -1 \end{pmatrix}$. Hence solve:

$$\begin{aligned} x - 2y + z &= 1 \\ 3x - 5y + 4z &= 4 \\ 2x - 2y + z &= -3 \end{aligned}$$

b) A matrix X is known to have eigenvalues 2, –1, and 1 and to have eigenvectors $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} -2 \\ -5 \\ -2 \end{pmatrix}$ corresponding, respectively, to these values. Find X.

3. a) State the Well-Ordering Principle.

b) Use induction to show that $7^n - 1$ is divisible by 6 for all $n \geq 1$.

c) (i) Calculate gcd(5310,2115).

(ii) Find integers n and m such that:

$$5310n + 2115m = \gcd(5310,2115).$$

4. a) Given a list of numbers that are between 0 and 1, written in decimal notation, describe a method for getting a number that is between 0 and 1 but is not contained in the list.

b) Show that every integer n can be factorised as a product of primes.

c) Find the prime factorisation of 1630827.

d) Prove that there are infinitely many prime numbers.

p.t.o.

5. a) Let $X = \{2,7,8,13,16,19,20\}$ and define a relation $\sim$ on X such that $x \sim y$ whenever $x - y$ is divisible by 5 (for $x, y \in X$). Show that $\sim$ is an equivalence relation and describe the partition of X into equivalence classes.

b) Let $X = \{(a,b) \mid a,b \text{ integers}, b \neq 0\}$ and define a relation $\sim$ on X as $(a,b) \sim (c,d)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation on X. (You may assume this.) Denote by $[(a,b)]$ the equivalence class of $(a,b) \in X$ and define addition of equivalence classes as follows:

$$[(a,b)] + [(c,d)] = [(ad + bc,\ bd)]$$

Show that this is independent of the choice of the representatives (a,b) and (c,d) within their equivalence classes.

c) Find the inverse of 10:

(i) in $\mathbf{Z}_{13}$

(ii) in $\mathbf{Z}_{17}$

(iii) Why does 10 not have an inverse in $\mathbf{Z}_{15}$?

6. a) Show that $a \in \mathbf{Z}_m$ has an inverse if and only if $\gcd(a,m) = 1$.

b) Use Euler's Theorem to find the remainder when 3^{340} is divided by 341.

c) Find the general solution of the following simultaneous congruences:

$x \equiv 8 \pmod{11}$ $\qquad$ $x \equiv -3 \pmod{13}$ $\qquad$ $x \equiv 4 \pmod{15}$

p.t.o.

7. a) (i) Find the quotient and the remainder when $x^4 + x^3 + 1$ is divided by $x^2 + x + 1$ in $\mathbf{Z}_7[x]$.

(ii) Find the irreducible factors of $x^3 + 6x^2 + 3x + 3$ in $\mathbf{Z}_{13}[x]$.

b) Write the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 8 & 10 & 3 & 12 & 6 & 16 & 9 & 1 & 13 & 15 & 14 & 5 & 2 & 4 & 7 & 11 \end{pmatrix}$$

(i) as a product of disjoint cycles,

(ii) as a product of transpositions.

(iii) Find the order and the sign of π.

c) Explain how to write a permutation $\pi \in S_n$ as a product of transpositions. What is the sign of a permutation in terms of the product of transpositions?

8. a) State and prove Lagrange's Theorem on the order of a subgroup of a finite group.

b) Write out the multiplication table for the group S_3 of all permutations of three points. What are the subgroups of S_3? Is S_3 isomorphic to the cyclic group of order 6? Give reasons.