OLLSCOIL NA HÉIREANN, GAILLIMH
THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

FIRST ENGINEERING & INFORMATION TECHNOLOGY EXAMINATION

MATHEMATICS [MA150]

MA151 — *CALCULUS*

PASS
First Paper

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Time allowed: *Three* hours.
Answer six questions.

1. a) Evaluate the following limits

(i) $$\lim_{x \to \infty} \frac{2x-1}{\sqrt{3x^2-1}+5}$$

(ii) $$\lim_{x \to -4} \frac{2x^2+9x+4}{x^2+x-12}$$

(iii) $$\lim_{x \to 0} \frac{\sin 3x}{\tan 5x}.$$

b) Let $f(x) = \dfrac{6-x^2+x}{x^2+4x+3}$

(i) Find all the asymptotes and intercepts of $f(x)$.
(ii) Find the intersection points of the horizontal asymptote and the graph of f, if any.
(iii) Find the values of x for which $f'(x) < 0$ and the values of x for which $f'(x) > 0$. Hence determine where $f(x)$ is increasing and where $f(x)$ is decreasing.
(iv) Sketch the graph of $f(x)$. **(DO NOT USE GRAPH PAPER.)**

P.T.O.

2. a) Use the $\in, \delta$ definition of a limit to prove that

$$\lim_{x \to -3} \frac{x^2 - 4}{3x + 6} = -5/3.$$

b) Find the value of the constants α and β for which the following function f is continuous at every point.

$$f(x) = \begin{cases} (x+1)^3 - 2 & \text{if } x \leq -1 \\ \alpha x + \beta & \text{if } -1 < x \leq 1 \\ 2x^2 - 3x + 1 & \text{if } x > 1 \end{cases}$$

For these values of α and β, is f differentiable at $x = 1$?

(c) Differentiate the following functions:

(i) $y = \dfrac{(x+4)^2 \cos x}{x^2 + 4}$

(ii) $y = (\ln x)^{\ln x}$ $(x > 1)$

(iii) $y = \coth^{-1} \sqrt{x^2 + 1}$

3. a) Show that, if a function f is defined on an open interval containing the point a and f is differentiable at a then f is continuous at a.

b) Oil from an uncapped oil well in the ocean is radiating outward in the form of a circular film on the surface of the water. If the radius of the circle is increasing at a rate of 2.5 meters per minute, how fast is the area of the oil film growing when the radius reaches 100 meters?

c) Use l'Hôpital's Rule to evaluate the limit

$$\lim_{x \to 0} \frac{2e^x - 2x - 2}{3x^2}$$

4. a) State the Mean Value Theorem. Deduce that, if f and g are differentiable functions such that $f'(x) = g'(x)$ for every x in the open interval (α, β) then there is a constant C such that $g(x) = h(x) + C$ for every x in (α, β).

b) A farmer has 900m of fencing with which she plans to enclose a rectangular pasture adjacent to a long existing wall. She plans to build one fence parallel to the wall, two to form the ends of the enclosure, and a fourth (parallel to the ends of the enclosure) to divide it equally. What is the maximum area that can be enclosed?

P.T.O.

5. a) State the Fundamental Theorem of the Calculus.

b) Use the Fundamental Theorem of the Calculus to calculate

$$\frac{d}{dx}\int_{\sqrt{x}}^{x} e^{t^2}\,dt.$$

c) Estimate $\int_0^{\pi/2} \sin x\,dx$ by computing the lower Riemann sum $L(\sin x, 2)$ and the upper Riemann sum U(sin x, 2) with 2 equal subintervals in each case.

Calculate also $L(\sin x, 3)$ and $U(\sin x, 2)$, and verify that

$$L(\sin x, 2) < L(\sin x, 3) < \int_0^{\pi/2} \sin x\,dx < U(\sin x, 3) < \mathrm{U}(\sin x, 2)$$

6. a) Evaluate **two** of the following integrals:

(i) $\int x^2\sqrt{x+2}\,dx;$

(ii) $\int \sin^4 x\,dx$

(iii) $\int \frac{5x-3}{x^2-2x-3}\,dx$

b) Establish the reduction formula:

$$\int \tan^n x\,dx = \frac{1}{\mathrm{n}-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \quad (n \neq 1)$$

and use it to evaluate $\int_0^{\pi/4} \tan^4 x\,dx$.

7. a) A hole of diameter a is bored symmetrically through the centre of a sphere of radius a. Show that the remaining volume is

$$\frac{\sqrt{3}}{2}\pi a^3.$$

b) Show that $\int_0^{\infty} \frac{dx}{1+e^x}$ converges and evaluate the integral.

P.T.O.

8. Solve the following differential equations:

(i) $x^2 \dfrac{dy}{dx} = xy - y^2$

(ii) $x \dfrac{dy}{dx} + 5y + 3x^4 = 0,\ x > 0$

(iii) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 10y = 0.$