GD1579

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THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

FIRST ENGINEERING & I.T. EXAMINATION

MATHEMATICS [MA150]

MA153 — *ALGEBRA*

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Time allowed: *Three* hours.
Answer six questions.

1. (a) Solve, if possible, the following system of linear equations in x, y, z, s, and t. Give the general solution.

$$\begin{aligned} x + 2y - 3z - 2s + 4t &= 1 \\ 2x + 5y - 8z - s + 6t &= 4 \\ x + 4y - 7z + 5s + 2t &= 8 \end{aligned}$$

(b) Determine the values of k so that the following system in x, y, and z has:

(i) a unique solution,
(ii) no solution,
(iii) infinitely many solutions.

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2 \end{aligned}$$

2. Let $A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{pmatrix}$.

(a) Find the adjoint A^* of A.
(b) Calculate AA^*.
(c) State the determinant of A and write down A^{-1}.
(d) Solve the matrix equation $AB = A^3 + 2A$ to find B.

p.t.o.

3. (a) Evaluate

$$\begin{vmatrix} -5 & 3 & 4 & -2 \\ 0 & 0 & 0 & 7 \\ -1 & 1 & -3 & 3 \\ 1 & -4 & 4 & 0 \end{vmatrix}$$

Hence, or otherwise, evaluate

$$\begin{vmatrix} 1 & -4 & 4 & 0 \\ 0 & 0 & 0 & 7 \\ -1 & 1 & -3 & 3 \\ -1 & 3/5 & 4/5 & -2/5 \end{vmatrix}.$$

(b) Show that $\begin{vmatrix} 2 & a & b \\ b & 2 & ab \\ a & ab & 2 \end{vmatrix} = 2^3 + (a-b-2)(a-b+2)ab.$

Hence find $\begin{vmatrix} 2 & 12 & 7 \\ 7 & 2 & 84 \\ 12 & 84 & 2 \end{vmatrix}$.

4. (a) Let $f: R^2 \to R^2$ be the function defined by

$$f(x,y) = (x+y, 2y).$$

(i) Show that f is a linear transformation.

(ii) Find the matrix of A of f; that is, find the 2×2 matrix A such that $(x,y)A = f(x,y)$ for all $x, y \in R$.

(b) Let f be the linear transformation that is rotation about the origin through an angle θ in R^2, and let g be the linear transformation that is rotation about the origin through an angle 2θ in R^2.

(i) State the matrix S of f, and the matrix T of g.

(ii) Show that $S^2 = T$. (Hint: Note that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$).

p.t.o.

5. (a) Use de Moivre's theorem to express $(1 + i)^{36}$ in the form $a + bi$ where a and b are real.

(b) Indicate on Argand diagrams the sets of points which satisfy

(i) $|z-i| = |z+i|$;

(ii) $2 \leq |z-4| < 4$;

(iii) $-\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$.

(c) Solve the equation $z^4 - 3z^3 - 12z - 16 = 0$, given that one root is of the form bi, where b is real.

6. (a) Let $z = \cos\theta + i\sin\theta$. Show that $z^n + z^{-n} = 2\cos n\theta$ and deduce the formula

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

(b) Write down the eighth roots of unity and hence express $z^4 + 1$ as a product of two real quadratic terms.

7. (a) Calculate the eigenvalues of the matrix $\begin{bmatrix} 8 & -4 & 2 \\ -2 & 1 & 1 \\ -12 & 6 & 0 \end{bmatrix}$.

(One of the eigenvalues is $\lambda = 0$.)
Hence, or otherwise, find a matrix P and a diagonal matrix D such that $P^{-1}AP$ is diagonal.

(b) Let B be an n x n matrix and let E be an invertible n x n matrix. Show that if λ is an eigenvalue of B then λ is also an eigenvalue of $E^{-1}BE$.

8. (a) Identify the following conic section and sketch its graph:

$$9x^2 - 4y^2 - 16y = 52.$$

(b) Find an orthogonal transformation which reduces the conic $2x^2 + 4xy + 5y^2 = 1$ to standard form. Sketch the conic, showing both sets of axes, and identify its type.