GX 1583

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THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

SECOND UNIVERSITY EXAMINATION

MATHEMATICS [MA280]

MA287 - ANALYSIS II

HONOURS

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Time allowed: *Two* hours.

Attempt *three* questions.

1. (a) Find the set of all possible values that can be assigned to $(1+i)^{3i}$.

(b) Give the definition of $f : \mathbb{C} \to \mathbb{C}$; $f(z) = \sqrt{z}$. Explain in your own words (and pictures) why f is not continuous at any point on the negative real axis.

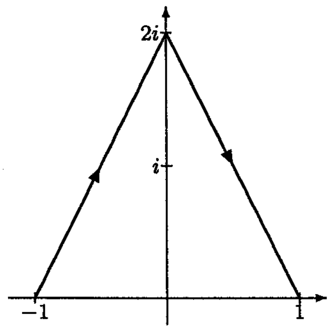
(c) Define what is meant by saying that, $g(z)$ is *differentiable* at $z = x_0 + iy_0$.
Prove that if the function $g : \mathbb{C} \to \mathbb{C}$; $g(x+iy) = u(x,y)+iv(x,y)$ is differentiable at $z = x_0+iy_0$ then

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

(d) The function $h : \mathbb{C} \to \mathbb{C}$; $h(x+iy) = x^2 - 2x - y^2 + iv(x,y)$ is differentiable for all $x + iy \in \mathbb{C}$ and $h(0) = 2i$. Find v.

p.t.o.

2. (a) Evaluate $\int_\gamma z^2 dz$, where γ^* is the path made up of the straight line segment joining -1 to $2i$ and the straight line segment joining $2i$ to 1; as shown in the picture below.



Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on the whole of $\mathbb{C}$, and let λ^* be the upper half of the clockwise unit circle. Explain, briefly, why $\int_\gamma f(z)dz$ has the same value as $\int_\lambda f(z)dz$.

(b) Give a proof of the following statement:

Suppose that η is a positively oriented simple closed path, that $\bar{D}(a,r) \subset I(\eta)$, and that g is holomorphic on an open set containing η^* and $I(\eta)$ except maybe at a. Then

$$\int_\eta g(z)dz = \int_{C(a,r)} g(z)dz$$

You may assume Cauchy's Theorem.

(c) Use the statement in part (b) to evaluate

$$\int_\eta \frac{1}{z-i} + \frac{1}{z+3i} dz$$

where $\eta : (-\pi, \pi] \to \mathbb{C}$; $\eta(t) = 4\cos(t) + 2i\sin(t)$.
Describe, briefly, how you made use of the statement in part (b) and justify your use of it.

p.t.o.

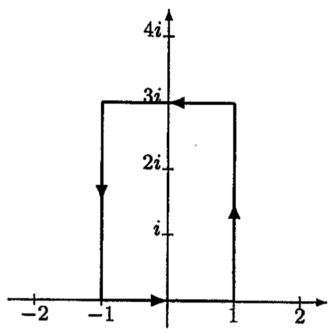
3. (a) Define the winding number, $n(\gamma, a)$, of a closed path γ about $a \notin \gamma^*$.
Draw a sketch of γ^* when

$$\gamma : [0, 7\pi] \to \mathbb{C};\ \gamma(t) = \begin{cases} (\pi + t)e^{it} & \text{if } 0 \leq t \leq 6\pi \\ 7\pi - 6(t - 6\pi) & \text{if } 6\pi \leq t \leq 7\pi \end{cases}$$

and write down $n(\gamma, 0)$. Hence, or otherwise, find $\int_\gamma \frac{1}{z} dz$.

(b) State Cauchy's integral formula.

(c) Let η^* be the oriented rectangle with vertices at $1 + 3i$, $-1 + 3i$, -1 and 1, taken in this order; as shown in the sketch below:



Evaluate: (i) $\int_\eta \frac{e^z}{z^2 - 4z + 5} dz$ and (ii) $\int_\eta \frac{z^2}{4z^2 + 4z + 5} dz$

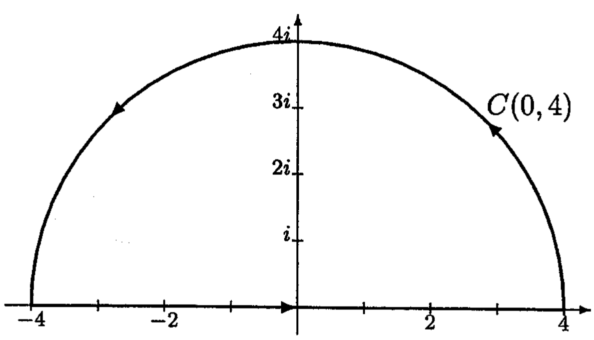
4. (a) State Laurent's Theorem.

(b) Find the Laurent Series for $f : \mathbb{C} \to \mathbb{C};\ f(z) = \frac{1}{z+1} + \frac{1}{z-2}$ which are valid for:
(i) $|z| < 1$ and (ii) $1 < |z| < 2$.

(c) Use Cauchy's Residue Theorem to evaluate:

(i) $\int_\gamma \frac{\sin(3z)}{z^2} dz$ where γ^* is the anti-clockwise unit circle.

(ii) $\int_\eta \frac{1}{z^2 + 9} dz$ where η^* is the path drawn below:



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