GX1585

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 1999 — HONOURS

THIRD ARTS and SCIENCE EXAMINATIONS

MATHEMATICS and COMPUTING STUDIES

MA385 and MA378 — Numerical Analysis

Professor J. Wiegold
Professor T. C. Hurley
Dr R. S. Dark

Time allowed: *three* hours.
Those taking one section should answer *three* questions.
Those taking both sections should answer *five* questions.

SECTION A — MA385

1. Consider the Initial Value Problem
$$y' = \frac{x+y}{2}, \quad y(0) = 1,$$
Describe the associated vector field, and sketch the solution of the problem.
Find approximations for $y(0.4)$, by applying the Modified Euler Method, first taking step size $h = 0.2$, and then $h = 0.1$.
Obtain an estimate for the error in the second approximation.

2. Suppose that $y(x)$ is the solution of the Initial Value Problem $y' = f(x, y)$, $y(x_0) = y_0$. Take $x_i = x_0 + ih$, and let y_i be an estimate for $\overline{y}_i = y(x_i)$ obtained using the formula $y_{i+1} = y_i + \Phi(x_i, y_i, h)h$. Show that if Φ satisfies a suitable Lipschitz Condition, then the error $\epsilon_i = y_i - \overline{y}_i$ satisfies the inequality
$$|\epsilon_{i+1}| < (1 + Mh)|\epsilon_i| + |\Phi(x_i, \overline{y}_i, h) - \Delta(x_i, \overline{y}_i, h)|\, h,$$
where M is a constant, and $\Delta(x_i, \overline{y}_i, h)$ is the slope of the chord from $(x_i, \overline{y}_i)$ to $(x_{i+1}, \overline{y}_{i+1})$.
Use Taylor's theorem to show that if $\Phi(x, y, h) = \lambda f(x, y) + \mu f(x + \alpha h, y + \beta k)$ with $k = f(x, y)h$, then
$$\Phi(x, y, h) - \Delta(x, y, h) = O(h^2)$$
provided $\lambda + \mu = 1$ and $\mu\alpha = \mu\beta = 1/2$.

3. (i) Explain how floating point numbers are stored in a computer, and show that the relative rounding error is bounded by the *precision* of the computer.

(ii) Give the definition of a *well-conditioned* function $\phi(a, b)$. Show that $\phi(a, b) = ab$ is well-conditioned, but give a numerical example for which $\phi(a, b) = a + b$ is ill-conditioned.

(iii) Give the definition of a *numerically stable* algorithm. Suppose that $y = -a + \sqrt{a^2 + b}$ with $a \gg b > 0$, and consider the algorithms which take $u = \sqrt{a^2 + b}$, and then:

(a) $y = -a + u$; (b) $y = \dfrac{b}{a + u}$.

Show that (b) is numerically stable, but (a) is not.

4. (i) Suppose that $x_{i+1} = g(x_i)$ $(i \geq 0)$, and $x_i \to c$ as $i \to \infty$, where $c = g(c)$. Prove that if $g'(c) = 0$, then the error $e_i = x_i - c$ satisfies the inequality

$$|e_{i+1}| \leq K|e_i|^2 \quad \text{for some constant } K.$$

Describe Newton's method for approximating a root of the equation $f(x) = 0$, and show that if it converges, then the convergence is quadratic.

(ii) Suppose that the vector function $\mathbf{u}(x) = (y(x), z(x))$ is a solution of the Initial Value Problem

$$\mathbf{u}' = (f(x, y, z),\ g(x, y, z)), \quad \mathbf{u}(x_0) = (y_9,\ z_0).$$

Describe carefully the Runge-Kutta method of order 4 for approximating $\mathbf{u}$, and explain how it can be applied to a second order Initial Value Problem

$$y'' = \phi(x, y, y'), \quad y(x_0) = y_0, \ y'(x_0) = \alpha.$$

SECTION B — MA378

5. Show that if $L(u) = -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2}$, then the bilinear form $\langle u, v\rangle_L = \displaystyle\iint_\Omega L(u)v\,dx\,dy$ is symmetric and positive definite on a suitable space D, consisting of functions which vanish on the boundary ∂D.

[You may assume Green's Theorem: $\displaystyle\iint_\Omega (p_x + q_y)\,dx\,dy = \int_{\partial\Omega} p\,dy - q\,dx$.]

Take grid points $P_0 = (0, 0)$, $P_1 = (1, 1)$, $P_2 = (2, 2)$, $P_3 = (3, 3)$, $P_4 = (3, 0)$, $P_5 = (0, 3)$, and let Ω be the square with vertices P_0, P_3, P_4, P_5. Find piecewise linear functions ϕ_1, ϕ_2 such that ϕ_i takes the value 1 at P_i, and vanishes at all the other points P_j. Then find an approximate solution of the Boundary Value Problem $L(u) = 2$, $u(x, y) = 0$ when $(x, y) \in \partial\Omega$, by solving the simultaneous equations

$$\langle\phi_1, \phi_j\rangle_L\lambda_1 + \langle\phi_2, \phi_j\rangle_L\lambda_2 = \langle 2, \phi_j\rangle \quad (j = 1,\ 2).$$

6. (i) Show that if $p(x)$ is a polynomial of degree n with $p(x_i) = f(x_i)$ for $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a, b]$, then for each value $x \in [a, b]$, there is a point $\xi \in [a, b]$ such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

(ii) The Trapezium Rule $Q(f) = \dfrac{b-a}{2}(f(a) + f(b))$ is used to approximate $I(f) = \displaystyle\int_a^b f(x)\,dx$. Assuming that $|f''(x)| < M$ for all $x \in [a, b]$, show that

$$|Q(f) - I(f)| < \frac{M}{12}(b-a)^3.$$

Use the Trapezium Rule to approximate $\displaystyle\int_0^{\pi/2} \sin x\,dx$, and find the error in the estimate.

7. Suppose that $p_0(x), p_1(x), \ldots, p_n(x)$ are monic polynomials, which are orthogonal with respect to the inner product $\langle f, g\rangle = \displaystyle\int_a^b f(x)g(x)\,dx$, and that p_r has degree r.

Show that if $x_0, x_1, \ldots, x_n$ are distinct, then the matrix P with entries $p_i(x_j)$ is non-singular.

Hence find numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that the formula $Q(f) = \displaystyle\sum_{i=0}^{n} \alpha_i f(x_i)$ has precision n.

Show also that $p_{n+1}(x) = (x - a_n)p_n(x) - b_n p_{n-1}(x)$, and find expressions for a_n and b_n.

8. (i) Show that if $x_i = x_0 + ih$, and if $f(x_i) = f_i$ for $i = n-2,\ n-1,\ n$, then

$$\int_{x_n}^{x_{n+1}} f(x)\,dx \quad \text{is approximated by} \quad (5f_{n-2} - 16f_{n-1} + 23f_n)\frac{h}{12}.$$

(ii) Consider the Initial Value Problem $y' = f(x, y)$, $y(x_0) = y_0$. Put $f_i = f(x_i, y_i)$, and show that if $y_{n+1} = y_n + (5f_{n-2} - 16f_{n-1} + 23f_n)h/12$, and $\epsilon_i = y_i - y(x_i)$ then under suitable conditions,

$$|\epsilon_{n+1}| < (1 + Ah)|\epsilon_n| + Bh^4.$$

[You may assume the formula in Question 6 (i).]