GX1587

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

COMPLEX VARIABLES (MA302)

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Time allowed: two hours.
Attempt *three* questions.

1. (a) Using de Moivre's theorem, find the fifth roots of -1.

   (b) Assuming that
   $$e^{iz} = \cos(z) + i\sin(z)$$
   and
   $$\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$
   hold for complex variables z and w, show that if $z = x + iy$ then
   $$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$
   What are the images of the lines $x = \frac{\pi}{4}$ and $y = 1$ under the map $z \mapsto \sin(z)$?

   (c) If $e^w = z$ for $z = re^{i\theta}$ $(r > 0)$ and $w = u + iv$ then find an expression for $\log(z) = w$. Show how to use this complex logarithm to define z^w for $z \neq 0$ and calculate i^i using the principal logarithm.

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2. Let f be a differentiable complex function and let $z = x + iy$.

   (a) State the Cauchy-Riemann equations which hold for

   $$f(z) = u(x, y) + iv(x, y).$$

   (b) If $f(z) = z^3 + z$, find u and v and show that they satisfy the Cauchy-Riemann equations.

   (c) Show that the function

   $$u(x, y) = x^2 - y^2 + x$$

   is harmonic, find a harmonic conjugate v for u and write $u + iv$ as a function of the complex variable z.

3. (a) Let C be a circle of radius 2 about the origin, travelling clockwise. Evaluate

   $$\oint_C \frac{1}{z} dz.$$

   (b) Find an antiderivative for the complex function $f(z) = z^2$.

   (c) State Cauchy's integral formula for a circle, explaining all terms in the formula. If C is a circle of radius 1 about the origin, travelling anticlockwise, use Cauchy's integral formula to evaluate

   $$\int_C \frac{1}{z^2 - 1} dz.$$

4. (a) Let

   $$f(z) = \frac{g(z)}{(z - a)^k}$$

   where g is a complex function differentiable in a disc of strictly positive radius centred at a. Define the *residue at a* of f. Calculate the residue at each pole of the function

   $$f(z) = \frac{1}{(z + i)^2(z - 3)}$$

   (b) State the residue theorem and use it to evaluate the real integral

   $$\int_{-\infty}^{\infty} \frac{1}{x^2 + 4} dx.$$