GX1591

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

B.Sc. EXAMINATION

MATHEMATICS

MA426 — *WAVELETS*

HONOURS

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Professor T. Hurley,
Dr. R. A. Ryan.

Time allowed: *Two* hours.
Full marks for *three* questions.

1. (a) Define the Discrete Time Fourier Transform (DTFT), $X(\omega)$, of a signal $x(n)$. Show that $X(\omega)$ is a 2π-periodic function, and that if $x(n)$ is real, then $|X(-\omega)| = |X(\omega)|$ for every ω.

   (b) Let $x(n) = (-1)^n$ for $-K \leq n \leq K$, and $x(n) = 0$ otherwise. Calculate the DTFT, $X(\omega)$, of $x(n)$, and sketch the graph of $X(\omega)$ for $-\pi \leq \omega \leq \pi$. How would you describe the frequency spectrum of $x(n)$?

   (c) Define the convolution product, $z = x * y$ and show that $Z(\omega) = X(\omega)Y(\omega)$

2. (a) What is a filter? Explain how one computes the impulse response and the frequency response of a filter.

   (b) Consider the moving difference filter, defined by

   $$y(n) = \frac{1}{2}(x(n) - x(n-1)).$$

   Find the impulse response, $h(n)$, and the frequency response, $H(\omega)$, and sketch the graph of $|H(\omega)|$. Why is this considered to be a high-pass filter?

   (c) Let $k = h * h$, where h is as in part (b). Write down the formula for the filter that has k as its impluse response and sketch the graph of $|K(\omega)|$.

p.t.o.

3. (a) Find the coefficients for the ideal low-pass filter that will stop all frequencies above a certain value α. What are the practical difficulties involved with the implementation of this filter?

   (b) A low-pass filter has non-zero coefficients $h(0), \ldots, h(N)$. Show that the filter whose non-zero coefficients are given by $k(n) = (-1)^n h(N-n)$ is a high-pass filter.

   (c) What is a filter bank? Describe the Haar filter bank and illustrate its operation by tracking the signal $(0, 2, 0, 1, 2, -1, -1, 0)$ as passes through the filter bank.

4. (a) Describe the Fast Wavelet Transform and apply it to the signal $(9, 7, 2, -2, 3, 7, 6, 8)$. Suppose all the wavelet coefficients that are smaller than 2 in absolute value are set to zero. Compute the inverse transform and compare it graphically with the original signal. Describe briefly how the Fast Wavelet Transform can be used in image compression.

   (b) Explain how the operations of dilation and translation are used to construct the Haar wavelet basis on $[0, 1]$. The function f is continuous and piecewise linear on $[0, 1]$ and satisfies $f(t) = 0$ on $[0, 1/2] \cup [5/8, 1]$ and $f(9/16) = 2$. Compute the Haar wavelet expansion of f up to level 3, and sketch this approximation to f. In what respects does the wavelet expansion of f differ from its Fourier expansion?

5. (a) Give the definition of a Multiresolution Analysis on $\mathbf{R}$. Illustrate the definition by reference to the Haar multiresolution analysis. Derive the Dilation Equation,

$$\varphi(t) = \sqrt{2} \sum_k c(k) \varphi(2t - k)$$

   for the scaling function, φ.

   (b) Show that the coefficients in the dilation equation satisfy

$$\sum_k c(k) c(k - 2m) = 0$$

   for every non-zero integer m. Deduce that it is not possible to have only three nonzero dilation coefficients, $c(0), c(1), c(2)$.

   (c) Explain how a wavelet basis can be generated from a multiresolution analysis.