GX1594

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THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

SECOND ENGINEERING EXAMINATION

MATHEMATICS [MA250]

Second Paper

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Time allowed: *three* hours.
Answer five questions in total but not more than *three questions from each section.*

SECTION A: *CALCULUS*

1. a) Evaluate

$$\iint_R y\, e^{x^2}\, dA$$

where

$$R = \{(x,y) : y^2 \le x \le 4, \quad 0 \le y \le 2\}.$$

b) By changing the order of integration evaluate

$$\int_0^1 \int_{y^{\frac{1}{3}}}^1 \cos(x^4)\, dx\, dy.$$

c) Let $R = \{(x,y) : x^2 + y^2 - 6x \le 0,\ y \ge 0\}$. Change the integral

$$\iint_R \frac{1}{\sqrt[4]{x^2+y^2}}\, dx\, dy$$

into an equivalent polar integral and then evaluate the polar integral.

p.t.o.

2. a) Define the complex functions e^z, $\cosh(z)$, and $\cos(z)$ as infinite series. Deduce that $\cosh(iz) = \cos(z)$, $\cosh(z) = ½(e^z + e^{-z})$ and $\cos(z) = ½(e^{iz} + e^{-iz})$.

b) Express e^z and $\cosh(z)$ in the form $u + iv$ where u and v are real. What is the periodicity, if any, of these functions?

c) Find the principal value of i^{-i}.

3. a) Let $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$ be a differentiable complex function. State (without proof) the Cauchy-Riemann equations for f. From these equations deduce that u and v are harmonic functions.

b) Show that $u(x,y) = 2x(y + 1)$ is harmonic and find a harmonic conjugate for u.

c) State an expression for $f'(z)$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ and show that if $f(z) = \sin(z)$, then $f'(z) = \cos(z)$.

4. a) Explain what is meant by a *pole of order n* and the *residue at a pole* of a complex function f. Calculate the residue at each pole of the function

$$f(z) = \frac{1}{(z+1)^2(z-i)}$$

b) Use the theory of residues to show that

$$\int_0^\infty \frac{\cos(2x)}{x^2+1}dx = \frac{\pi e^{-2}}{2}.$$

p.t.o.

SECTION B: *LINEAR ALGEBRA*

1. a) State the Cauchy-Schwarz-Bunjakowski Inequality and use it to prove the Triangle Inequality

$$\|a+b\| \leq \|a\| + \|b\|,$$

where a, b are vectors in a vector space with inner product < , > and norm satisfying

$$\|v\|^2 = \langle v, v \rangle.$$

b) What is an orthonormal basis?
Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of a vector space V. If $v \in V$, explain how to find $\alpha_1, \ldots \alpha_n$ such that $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$.

c) Show that the following vectors form an orthonormal basis for $\mathbf{R}^3$

$$\left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{-1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right).$$

2. a) Show that the functions $1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos nt, \sin nt, \ldots$ constitute an orthogonal set on the interval $[-\pi, \pi]$. Find the orthogonal projection of the function $f(t) = t + 5$ onto the space spanned by the functions 1 and $\cos t$.

b) Describe the Gram-Schmidt Process and use it to find an orthornormal basis for the subspace of $\mathbf{R}^4$ defined by

$$x_1 - x_2 + 3x_3 + 2x_4 = 0,$$

by converting the basis $\{(1,4,1,0), (0,3,1,0), (-1,1,0,1)\}$ into an orthonormal basis.

3. a) Let A be a real symmetric matrix. Prove that the eigenvectors corresponding to distinct eigenvalues of A are orthogonal to each other.

b) Determine the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 3 \end{pmatrix}$$

and hence or otherwise, find an orthogonal matrix P such that $P^t AP$ is a diagonal matrix.

4. a) Show that the orthogonal projection of a vector X in $\mathbf{R}^n$ onto a subspace W is a given by

$$A(A^T A)^{-1} A^T X$$

where the columns of the matrix A form a basis for W.

b) Find the least squares approximate solution to the following overdetermined system of equations:

$$\begin{array}{rcrcr} x & + & y & = & 4 \\ x & - & y & = & 1 \\ 2x & + & y & = & 8 \end{array}$$