GX1601

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

MATHEMATICS

[MA344 — GROUPS II]

HONOURS

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Time allowed: *two* hours.
Answer *three* questions.

1. Let G be a finite group acting on a finite set X. For each $g \in G$ and $x \in X$, define $x\theta(g) \in X$ by

$$x\theta(g) = x.g$$

(a) Prove that (i) $\theta(g) \in \mathrm{Sym}(X)$; (ii) θ is a group homomorphism.

(b) For $x, g \in G$, define $x.g$ to be $g^{-1}x$. Show that this defines a right action of G on itself, and find $\ker\theta$ in this case. Deduce that S_4 contains subgroups isomorphic to $C_2 \times C_2$ and C_4.

(c) Let G act on itself by conjugation. With θ as defined at the beginning, prove that if $\mathrm{im}\,\theta$ is a cyclic group then G is abelian.

2. (a) Let G be a finite group acting on a finite set X.

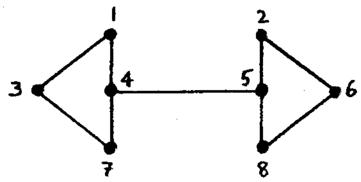
(i) For $x \in X$ and $g \in G$, define the orbit xG, the stabiliser G_x, and the set of fixed points $\mathrm{fix}\,g$.

(ii) Use the fact that $|G| = |xG||G_x|$ to prove that the number of distinct orbits is

$$\frac{1}{|G|}\sum_{g\in G} |\mathrm{fix}\,g|.$$

Also show that if $|G| = p^n$ for some prime p not dividing $|X|$, then there is $x \in X$ such that $xg = x$ for all $g \in G$.

(b) Let $\mathcal{G}$ be the graph



and let $G = \mathrm{Aut}(\mathcal{G})$.

(i) Find the distinct orbits in the vertex set of $\mathcal{G}$ under action of G.

(ii) Explain why $G_1 \subseteq G_3 \cap G_4 \cap G_7$. Hence determine G_1 and $|G|$.

(c) Find the number of essentially distinct colourings of the vertices of the graph $\mathcal{G}$ in (b), given the colours green, white and orange from which to choose.

3. (a) Let G be a group of order $p^k m$, where p is a prime not dividing m.

(i) State the existence, uniqueness and arithmetic parts of Sylow's Theorem.

(ii) Assuming the uniqueness part of Sylow's Theorem, prove that the number of Sylow p-subgroups of G divides m. Also explain why G has a unique Sylow p-subgroup if and only if it has a normal Sylow p-subgroup.

(b) Show that G is a finite abelian simple group if and only if G is cyclic of prime order. Describe the simple groups of prime-power order. Give (without proof) an example of a nonabelian finite simple group.

(c) Prove that there is no simple group of order (i) 48, (ii) p^2q, where p and q are distinct primes.

4. Let $I = \{a, b\}$ and denote by $\mathbb{N}$ the set of non-negative integers.

(a) (i) Define the free monoid I^*.

(ii) Show that the function $f: I^* \to \mathbb{N}$ defined by $f(w) =$ the number of a's in $w \in I^*$, is a homomorphism of monoids (the operation in $\mathbb{N}$ is addition).

(iii) Let R be the relation on I^* defined by

$$v\,R\,w \Leftrightarrow f(v) = f(w).$$

Show that R is a congruence relation. Describe the quotient monoid I^*/R and prove that it is isomorphic to $\mathbb{N}$.

(b) Suppose I^* acts on $\{0, 1, 2\}$ by

$$0a = 1a = 2a = 0, \quad 0b = 1, \quad 1b = 2b = 2.$$

Give the diagram of this action and determine the languages

$$L(0,2) = \{w \in I^* \mid 0w = 2\}$$

$$[ba] = \{w \in I^* \mid xw = xba \text{ for } x = 0, 1, 2\}.$$