GX1602

SUMMER 1999

ALGEBRAIC TOPOLOGY [MA404]

(ERASMUS STUDENTS)
Time allowed: three hours.
Full marks for *five* questions.

A. CHRISTOFIDES.

1. Let f be a path from x_1 to x_2 and g be a path from x_2 to x_3, in a topological space X. Define the operation $f \star g$ and explain briefly why it is not associative.

Given topological spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps from X to Y.

(a) Let $I = [0,1]$. Show that for any X, the set $[X, I]$ has a single element.

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

2. Let α be a path from x_0 to x_1 in a topological space X. Let

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

be defined by the equation

$$\hat{\alpha}([f]) = [\tilde{\alpha}] \star [f] \star [\alpha],$$

where $\tilde{\alpha}$ denotes the reverse path of α. Show that $\hat{\alpha}$ is a group homomorphism.

Let $A \subset X$ and let $r : X \to A$ be a retraction. Given $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

Let A be a subset of $\mathbf{R}^n$; let $h : (A, a_0) \to (Y, y_0)$. Show that if h is extendable to a continuous map of $\mathbf{R}^n$ into Y, then h_* is the zero homomorphism.

3. Define a covering map $p : X \to Y$ between topological spaces X, Y. Give an outline of the proof that the fundamental group of the circle is infinite cyclic.

4. Define an inessential map $h : X \to Y$ between topological spaces. Give an outline of the proof of the following result:

Given a nonvanishing vector field on the unit disc $\mathbf{B}^2$, there exists a point of $\mathbf{S}^1$ where the vector field points directly inward and a point of $\mathbf{S}^1$ where it points directly outward.

Deduce Brouwer's fixed point theorem for the disc.

p.t.o.

5. Give the following definitions:

(a) The standard n-simplex Δ^n,

(b) a singular n-simplex in a topological space X.

(c) a singular n-chain in X

(d) the boundary operator $\partial_n : S_n(X) \to S_{n-1}(X)$

For all $n \geq 0$, show that $\partial_n \partial_{n+1} = 0$.

Define the nth homology group $H_n(X)$ of the space X and show that

$$H_n : \mathbf{Top} \to \mathbf{Ab}$$

is a functor.

6. Assume that $f, g : X \to Y$ are continuous maps and that there are homomorphisms $P_n : S_n(X) \to S_{n+1}(Y)$ with

$$f_* - g_* = \partial'_{n+1} P_n + P_{n-1} \partial_n.$$

Prove that, for all $n \geq 0$, $H_n(f) = H_n(g)$.

7. Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups. Show that $iA \cong A$ and $B/iA \cong C$ via $b + iA \mapsto pb$.

If

$$\cdots \to C_{n+1} \to A_n \xrightarrow{h_n} B_n \to C_n \to A_{n-1} \xrightarrow{h_{n-1}} B_{n-1} \to C_{n-1} \to \cdots$$

is exact and every third arrow $h_n : A_n \to B_n$ is an isomorphism, then $C_n = 0$ for all n.

If $0 \to A \to B \to C \to 0$ is a short exact sequence of finite dimensional vector spaces over $\mathbf{R}$, show that

$$\text{dimension}(B) = \text{dimension}(A) + \text{dimension}(C).$$

8. Define the Mayer-Vietoris exact sequence concerning the homology groups $H_n(X), H_n(X_1), H_n(X_2)$, $H_n(X_1 \cap X_2)$, where X_1 and X_2 are subspaces of a topological space X with $X = X_1^\circ \cup X_1^\circ$.

Use this sequence to obtain the homology groups or the spere $\mathbf{S}^n$ for $n > 0$.