GX1603

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

SECOND ENGINEERING EXAMINATION

MATHEMATICS [MA250]

MA251 — CALCULUS

PASS
First Paper

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Time allowed: *Three* hours.
Credit will be given for the best five questions.

This paper is available ONLY to candidates who are *repeating* the examination from a previous year.

1. (a) Show by approaching the origin along two different paths that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

(b) The positions of two airplanes at time t are given by

$$x = 2t-1, \qquad y = 1-t, \qquad z = 3t$$

and

$$x = t+1, \qquad y = 2t, \qquad z = 3-2t.$$

(i) Do the paths of the airplanes intersect?
(ii) Do the airplanes collide?

(c) Find the equation of the plane that contains the points $(0,0,0,)$, $(1,1,1)$ and $(-1,1,1)$.

2. Let

$$\vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + (4t)\vec{k}.$$

Find

(i) the unit tangent vector $(\vec{T})$ at $t=\pi$;
(ii) the principal unit normal vector $(\vec{N})$ at $t=\pi$;
(iii) the curvature (κ) at $t=\pi$;
(iv) the binormal vector $(\vec{B})$ at $t=\pi$;
(v) the equation of the osculating plane at $t=\pi$;
(vi) the torsion at $t=\pi$.

p.t.o.

3. (a) Let $u = u(x, y)$ where $x = st$ and $y = \frac{1}{2}(s^2 - t^2)$.
Use the chain rule to prove that:

$$\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 = (s^2 + t^2)\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right).$$

(b) Find all local maxima, local minima and saddle points for:

$$f(x, y) = \frac{x^2 y}{2} - 2y + 2y^2 + 5.$$

(c) Find the equation of the tangent plane to the surface $x^2 + 2y^2 + z^2 - 10 = 0$ at the point $(1, 0, 3)$.

4. (a) Suppose the temperature at a point $P(x, y, z)$ is given by

$$T = 4x^2 - y^2 + 16z^2.$$

(i) Find the rate of change of T at the point $P(4, -2, 1)$ in the direction of the vector $2\vec{i} + 6\vec{j} - 3\vec{k}$.
(ii) In what direction does T increase most rapidly?

(b) Find the minimum distance from $(3, -3, 1)$ to the paraboloid $z = x^2 + y^2$.

5. (a) Evaluate

$$\int\int_R y\, e^{x^2}\, dA$$

where

$$R = \{(x, y) : y^2 \leq x \leq 4, \quad 0 \leq y \leq 2\}.$$

(b) By changing the order of integration evaluate

$$\int_0^1 \int_{y^{1/3}}^1 \cos(x^4)\; dx\, dy.$$

(c) Let $R = \{(x, y) : x^2 + y^2 - 6x \leq 0,\ y \geq 0\}$.
Change the integral

$$\int_R \int \frac{1}{\sqrt{x^2 + y^2}}\, dx\, dy$$

into an equivalent polar integral and then evaluate the polar integral.

p.t.o

6. (a) Define the complex functions e^z, $\cosh z$ and $\cos(z)$ as infinite series.
Deduce that $\cosh(iz) = \cos(z) = \frac{1}{2}(e^z + e^{-z})$ and $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$.

(b) Express e^z and $\cosh(z)$ in the form $u + iv$ where u and v are real. What is the periodicity, if any, of these functions?

(c) Find the principal value of i^{-i}.

7. (a) Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ be a differentiable complex function. State (without proof) the Cauchy-Riemann equations for f. From these equations deduce that u and v are harmonic functions.

(b) Show that $u(x, y) = 2x(y + 1)$ is harmonic and find a harmonic conjugate for u.

(c) State an expression for $f'(z)$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ and show that if $f(z) = \sin(z)$ then $f'(z) = \cos(z)$.

8. (a) Explain what is meant by a *pole of order n* and the *residue at a pole* of a complex function f. Calculate the residue at each pole of the function

$$f(z) = \frac{1}{(z+1)^2(z-i)}.$$

(b) Use the theory of residues to show that

$$\int_0^\infty \frac{\cos(2x)}{x^2+1}\,dx = \frac{\pi e^{-2}}{2}.$$