GX1607

OLLSCOIL NA hÉIREANN, GAILLIMH
THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

SECOND ENGINEERING EXAMINATION

MATHEMATICS [MA250]

MA253 — *ALGEBRA*

PASS
Second Paper

Professor D.H. Armitage
Professor T.C. Hurley
Dr. R. Ryan
Dr. J. Ward

Time allowed: *Three* hours.
Credit will be given for the best five questions.

This paper is available ONLY to candidates who are *repeating* the examination from a previous year.

1. (a) What is meant by saying that a subset, V, of $\mathbf{R}^n$ is a *subspace* of $\mathbf{R}^n$. Show that the following set is *not* a subspace of $\mathbf{R}^4$:

$$V = \{(x_1, x_2, x_3, x_4) \quad : \quad x_1 + 2x_2 + x_3x_4 = 0\}.$$

(b) What is a *spanning set*? Find a spanning set for the solution space of the equation

$$x_1 - x_2 + x_3 + 2x_4 - 3x_5 = 0.$$

(c) What is meant by saying that a set of vectors is *linearly independent*? Show that the set consisting of the functions e^x, e^{2x}, e^{3x} is linearly independent.

2. (a) What is a *basis*? Determine whether the following vectors form a basis for $\mathbf{R}^3$:

$$(1, -3, 1), \qquad (0, 2, -1), \qquad (2, -4, 1).$$

(b) For the bases B_1 and B_2 given below, find the change of basis matrix that converts B_1-coordinates into B_2-coordinates:

$$B_1 : \{(1, 2, 0), \quad (2, 0, 3), \quad (2, 3, 1)\}$$
$$B_2 : \{(1, 0, 1), \quad (1, 1, 0), \quad (0, 1, 1)\}$$

p.t.o.

3. (a) What is a *linear transformation*? A mapping, T, from $\mathbf{R}^4$ into $\mathbf{R}^3$ is defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2, 2x_3, x_1 x_4).$$

Show that T is not linear.

(b) What are the *kernel* and the *image* of a linear transformation? The linear transformation, L, is given by the matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \\ 0 & 0 & 2 \\ 2 & -1 & -2 \end{pmatrix}.$$

Find a basis for the kernel of L and a basis for the image of L. Does the vector $(3, 2, 0, 1)$ belong to the image of L?

4. (a) What are the *eigenvalues* and the *eigenvectors* of an $n \times n$ matrix A? If A has n linearly independent eigenvectors, show that there is an invertible matrix, E, and a diagonal matrix, D, such that

$$E^{-1}AE = D.$$

(b) Find the matrices E and D when

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}.$$

5. (a) State the Cauchy-Schwarz-Bunjakowski Inequality and use it to prove the Triangle Inequality

$$||a + b|| \leq ||a|| + ||b||.$$

(b) What is an orthonormal basis?
Let $\{v_1, ..., v_n\}$ be an orthornormal basis of a vector space V. If $v \in V$, explain how to find $\alpha_1, ..., \alpha_n$ such that $v = \alpha_1 v_1 + ... + \alpha_n v_n$.

(c) Show that the following vectors form an orthonormal basis for $\mathbf{R}^3$:

$$\left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \quad \left(\frac{-1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right), \quad \left(\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$$

p.t.o

6. (a) Show that the functions $1, \cos t, \sin t, \cos 2t, \sin 2t, ..., \cos nt, \sin nt, ...$ constitute an orthogonal set on the interval $[-\pi, \pi]$. Find the orthogonal projection of the function $f(t) = t + 5$ onto the space spanned by the functions 1 and $\cos t$.

(b) Describe the Gram-Schmidt Process and use it to find an orthonormal basis for the subspace of $\mathbf{R}^4$ defined by

$$x_1 - x_2 + 3x_3 + 2x_4 = 0,$$

by converting the basis $\{(1,4,1,0), (0,3,1,0), (-1,1,0,1)\}$ into an orthonormal basis.

7. (a) Let A be a real symmetric matrix. Prove that the eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

(b) Determine the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 3 \end{pmatrix}$$

and hence or otherwise, find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix.

8. (a) Show that the orthogonal projection of a vector X in $\mathbf{R}^n$ onto a subspace W is given by

$$A(A^T A)^{-1} A^T X$$

where the columns of the matrix A form a basis for W.

(b) Find the least squares approximate solution to the following overdetermined system of equations:

$$\begin{aligned} x + y &= 4 \\ x - y &= 1 \\ 2x + y &= 8 \end{aligned}$$